THE UNIVERSITY OF SYDNEY
Business School

Differentiation

Mathematics Help Sheet

The University of Sydney Business School

Introduction

Differentiation is a branch of calculus that involves finding the rate of change of one variable with respect to another variable. In practice, this commonly involves finding the rate of change of a curve (generally a two-variate function that can be represented on a Cartesian plane).

You **differentiate** a function in order to find its **derivative,** which is a measure of how that function (the output) changes as its input(s) change(s).

A useful concept in differentiation is the **tangent** which is a straight line that just touches a curve at any given point. The slope of a tangent at any point is the measure of the rate of change of a curve at that particular point.

Differentiation Notation

When given a variable (y) expressed as a function of another variable (x) such that $y = f(x)$, we can **differentiate y with respect to x.** In terms of mathematical notation, the derivative (or the first derivative) of the general function $y = f(x)$ can be expressed as:

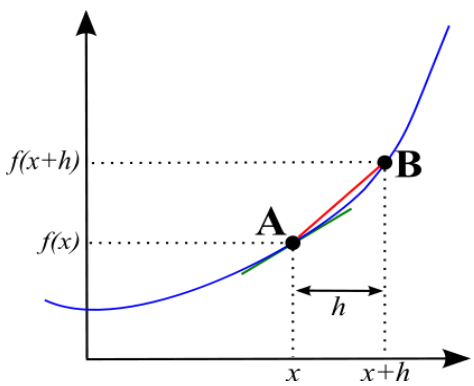
$$\frac{dy}{dx} \;\; \textbf{or} \;\; \frac{d}{dx}y \;\; \textbf{or} \;\; f'(x) \;\; \textbf{or} \;\; \frac{d}{dx}f(x) \;\; \textbf{or} \;\; y'$$

It is important to note that $\frac{dy}{dx}$ does not represent a fraction such that you can cancel out terms in the numerator or denominator.

Differentiation by First Principles

A formal statement of differentiation is as follows (in reference to the following graph):

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



That is, the first derivative of the function at point x is given by the slope (or the rise over run) of the curve as the distance between the two points (A and B) goes to 0 (i.e. they merge into a single point).

While it is still possible to use this formal statement in order to calculate derivatives, it is tedious and seldom used in practice. The following sections will introduce to you the rules of differentiating different types of functions.

Constants

The derivative of any constant is simply 0 since the value of the constant never changes. That is,

$$\frac{d}{dx}a = 0$$

Where a is any constant.

Polynomial Functions

A polynomial function is a function that contains only non-negative integer powers of a variable such as a quadratic or a cubic function, e.g. $y = x^2$ and $y = 3x^3$, respectively.

Where $y = ax^n$,

$$\frac{dy}{dx} = nax^{n-1}$$

That is, the new coefficient is the product of the original exponent and original coefficient, and the new exponent is the original exponent subtracted by 1.

For example,

$$f(x) = 2 + 3x^5$$

$$f'(x) = 15x^4$$

Note that in the preceding example, the constant disappears after differentiation, since the derivative of a constant is always 0.

Exponential Functions

An exponential function possesses a value that is raised to the power which is or contains the variable of interest, that is, it possesses the general form $y = a^x$. Where the base value is the constant “e”, there are special rules which exist for differentiating exponential functions.

Where $y = e^{f(x)}$,

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

That is, the derivative of the exponent becomes the new coefficient of "e", and the original expression with base "e" remains the same.

For example,

$$y = 2e^{4x^2}$$

$$\frac{dy}{dx} = 2(8x)e^{4x^2}$$

$$\frac{dy}{dx} = 16xe^{4x^2}$$

Note: for an explanation of exponential functions in general, please refer to the "Exponentials and Logarithms" help sheet.

Logarithmic Functions

A logarithmic function is a function that possesses the general form $y = log_a(x)$. When the base of the log is "e", i.e. we are dealing with $\ln(f(x))$, there are special rules for differentiating logarithmic functions.

Where $y = log_e(x)$,

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

That is, the derivative of the log consists of:

1) The derivative of the function within the log to form the **numerator**, and
2) The original log function to form the **denominator**

For example,

$$y = \ln(2x^2 + 5)$$

$$\frac{dy}{dx} = \frac{4x}{2x^2 + 5}$$

A harder example involving both an exponential function and logarithm,

$$y = \ln(e^{2x} + 2)$$

$$\frac{dy}{dx} = \frac{2e^{2x}}{e^{2x} + 2}$$

Note: for an explanation of logarithmic functions in general, please refer to the "Exponentials and Logarithms" help sheet.

Further Rules

The Chain Rule

The Chain rule is used where there is a function within a function. For example, where the functional form is $y = G(f(x))$,

$$\frac{dy}{dx} = G'\big(f(x)\big)f'(x)$$

The derivative of the above function is a product of the derivative of the inner function ($f(x)$) and the derivate of the outer function ($G(x)$).

For example,

$$y = [f(x)]^n$$

$$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$$

For example,

$$y = [x^2]^3$$

$$\frac{dy}{dx} = 3x[x^2]^2$$

The Product Rule

The product rule is used when you need to differentiate the product of two functions. Where $y = uv$,

$$\frac{dy}{dx} = uv' + vu'$$

The derivative is the sum of:

1) The first term multiplied by the derivative of the second term
2) The second term multiplied by the derivative of the first term

For example,

$$y = (x^2 + 12)(3x^3 + x)$$

$$\frac{dy}{dx} = (x^2 + 12)(9x^2 + 1) + 2x(3x^3 + x)$$

The Quotient Rule

The quotient rule is used when you have one function divided by another function.

Where $y = \frac{u}{v}$,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

The derivative has a numerator which consists:

1) The denominator of the original function multiplied by the derivative of the numerator of the original function, subtracted by
2) The numerator of the original function multiplied by the derivative of the denominator of the original function.

The derivative has a denominator which consists:

1) The denominator of the original function squared.

For example,

$$y = \frac{x^3 + 8}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2(3x^2) - (x^3 + 8)2x}{x^4}$$

$$\frac{dy}{dx} = \frac{x^3 - 16}{x^3}$$

Finding Maxima and Minima

In the Business School, you may be asked to find the maximum profit or minimum cost of a profit function. The easiest and quickest way to do this is to find the derivative of a function, and equate it to zero. This is because at a rate of change of zero, the tangent is horizontal and should correspond to the point where profits have reached their highest, or where costs have reached their lowest.

Consider for example, a business which has the following profit function,

$$\pi = pq - 10q$$

Where,

$$p = 30 - q$$

Thus,

$$\pi = (30 - q)q - 10q$$

$$\pi = 20q - q^2$$

Differentiating profit with respect to quantity,

$$\frac{d\pi}{dq} = 20 - 2q$$

Setting the first derivative to zero,

$$2q = 20$$

$$q = 10$$

Thus, the quantity that maximises profit is 10.

The Second Derivative

To check whether a result is a maximum or a minimum (or neither), you can use the second derivative. To find the second derivative, you simply differentiate the first derivative as you did the original function.

In the preceding example,

$$\frac{d\pi}{dq} = 20 - 2q$$

Differentiating again,

$$\frac{d^2\pi}{dq^2} = -2$$

Thus, the second derivative is negative which indicates a maximum.

The rules can be summarised as follows:

- If the second derivative is **positive**, you have a **minimum**
- If the second derivative is **negative**, you have a **maximum**
- If the second derivative is **0**, then you have **neither**