THE UNIVERSITY OF SYDNEY
Business School

Exponentials and Logarithms

Mathematics Help Sheet

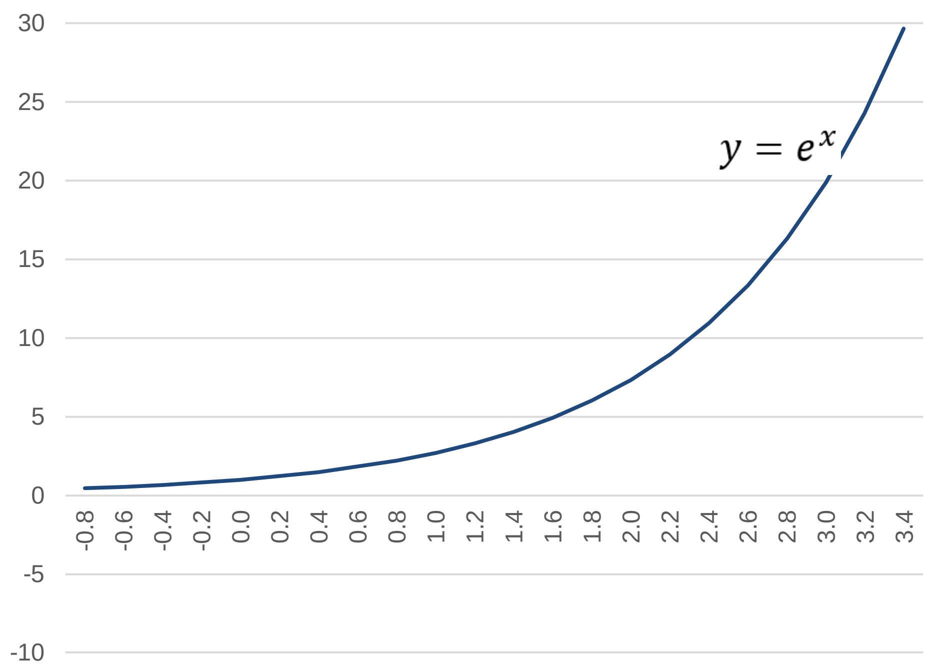
The University of Sydney Business School

Exponentials

Exponential Functions

An exponential function is a function that possesses the general form of $y = a^x$ where "a" is any constant greater than one. The graph of an exponential function will increase at an increasing rate as x increases, and it will also have an **asymptote** at the x-axis, that is, it will approach the x-axis but never touch it.

Consider the graph of an exponential function below,



The above graph represents the function $y = e^x$ where e is a constant equivalent to approximately 2.718.

Other examples of exponential functions include,

$$y = 10^x$$

$$y = 5^{3x}$$

$$y = 5^{2+x}$$

Index Laws

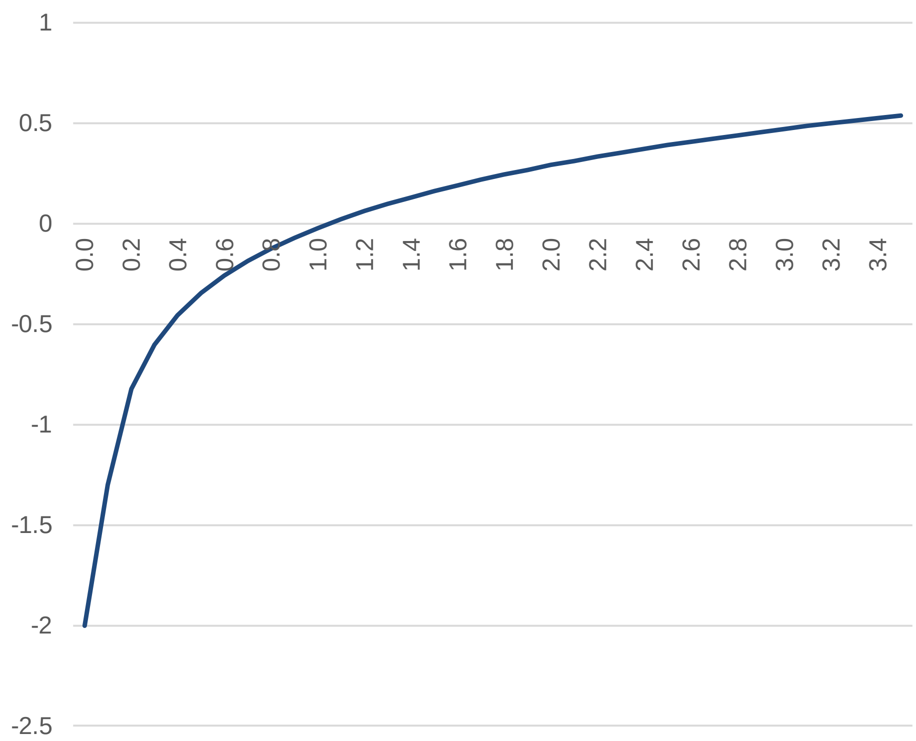
The below are rules and examples for using exponentials. Note that a, b, x and y can be any numbers.

	Rule	Example
1.	$x^0 = 1, \ for\ all\ x \neq 0$	$5^0 = 1$
2.	$x^a + x^b = x^{a+b}$	$x^5 + x^2 = x^7$
3.	$\frac{x^a}{x^b} = x^{a-b}$	$\frac{x^8}{x^3} = x^5$
4.	$(x^a)^b = x^{ab}$	$(x^2)^3 = x^6$
5.	$(xy)^a = x^a y^a$	$(xz)^3 = x^3 z^3$
6.	$x^{-1} = \frac{1}{x}$	$x^{-4} = \frac{1}{x^4}$
7.	$ax^{-b} = \frac{a}{x^b}$	$10x^{-2} = \frac{10}{x^2}$
8.	$x^{\frac{1}{a}} = \sqrt[a]{x}$	$x^{\frac{1}{3}} = \sqrt[3]{x}$
9.	$x^{\frac{a}{b}} = \sqrt[b]{x^a}$	$x^{\frac{3}{5}} = \sqrt[5]{x^3}$

Logarithms

A logarithmic function is a function that possesses the general form $y = log_a(x)$. Logarithmic functions increase infinitely but at a decreasing rate, as x increases. The function has an asymptote at the vertical axis, meaning it approaches, but never touches, the y-axis. This is because you cannot take a logarithm of a negative number, and hence the graph only exists for positive values of x.

Consider the graph of a logarithmic function below,



The function graphed above is $y = log_{10}(x)$, which can be read as "y is equal to log to the base 10 of x." Quite literally, the logarithm is the numerical quantity representing the number of times a constant (the base, which is 10 here) needs to be multiplied by itself in order to arrive at some other quantity (x).

Consider for example, when $x = 100$, such that,

$$y = log_{10}(100)$$

The logarithm is simply the number of times 10 needs to be multiplied by itself in order to reach 100, which is 2. Hence,

$$log_{10}(100) = 2$$

The Relationship between Exponentials and Logarithms

To understand a logarithm, you can think of it as the **inverse** of an exponential function. While an exponential function such as $x = 5^y$ tells you what you get when you multiply 5 by itself y times, the corresponding logarithm, $y = \log_5(x)$, asks the opposite question: how many times do you have to multiply 5 by itself in order to get x?

The relationship is as follows,

$$If \quad x = a^y, \quad then \;\; y = \log_\mathrm{a}(x)$$

For example,

$$8 = 2^3$$

$$3 = \log_2(8)$$

The Natural Log

The most commonly used base for logarithms is the base e, which is a constant approximately equal to 2.718. This is called the **natural logarithm** and is written using an "$\ln$".

That is,

$$\log_e(x) = \ln(x)$$

If you encounter a log without a base such as $\log(x)$, it is generally assumed that it is a **natural log**. The name is derived from its relationship with the natural exponential function, $y = e^x$, originally used to describe continuous growth in nature.

The reason that we use the base e most commonly is because it has some practical properties which allow for ease of calculation, in particular, with respect to differential calculus. Some of these properties are explored in the "Differentiation" help sheet.

Log Rules

	Rule	Example
1.	$\log(x^y) = ylog(x)$	$\log(100^5) = 5log(100)$
2.	$\log(xy) = \log(x) + \log(y)$	$\log(abc) = \log(a) + \log(b) + \log(c)$
3.	$\log\left(\frac{\mathrm{x}}{\mathrm{y}}\right) = \log(xy^{-1})$ $= \log(x) + \log(y^{-1})$ $= \log(x) - \log(y)$	$\log\left(\frac{8}{w}\right) = \log(8) - \log(w)$
4.	$\log(1) = 0$	
5.	$\log_{\mathrm{a}}(b) = \frac{\log_c(b)}{\log_c(a)}$	$\log_{10}(100) = \frac{\log_2(100)}{\log_2(10)}$
6.	$a^{\log_{\mathrm{a}}(x)} = x$	$e^{\ln(x)} = x$

Using Logarithms to Solve Exponential Functions

Logs are useful for solving equations where the unknown variable is a power rather than a base. Consider the following,

$$125 = 5^x$$

You can take the log of both sides of the equation,

$$\ln(125) = \ln(5^x)$$

$$\ln(125) = x\ln(5)$$

$$x = \frac{\ln(125)}{\ln(5)}$$

$$x = 3$$