THE UNIVERSITY OF SYDNEY
Business School

Matrix Algebra

Mathematics Help Sheet

The University of Sydney Business School

Introduction

Definitions and Notation

A matrix is a rectangular array (or arrangement) of elements that possesses the general form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NK} \end{bmatrix}$$

A matrix itself can be denoted as a single capital letter such as "**A**", as seen above. In general form, each element within a matrix is denoted as a lower case letter with a **subscript indicating the row,** followed by a **subscript indicating the column.**

For example, a 2 × 2 matrix looks like this:

$$A = \begin{bmatrix} 3 & 9 \\ 7 & 5 \end{bmatrix}$$

A 3 × 3 matrix looks like this:

$$A = \begin{bmatrix} 8 & 4 & 8 \\ 4 & 0 & 8 \\ 1 & 6 & 5 \end{bmatrix}$$

The above two matrices are called square matrices because they possess the same number of rows and columns.

The Identity Matrix

The identity matrix is a square matrix of any dimension with the value 1 forming the elements running diagonally throughout the matrix from top left to bottom right. It is the matrix equivalent of the number **one** such that any matrix multiplied by the identity matrix results in itself.

An identity matrix possesses the general form:

$$I_N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix Addition and Subtraction

To add two matrices, add each element with its corresponding element in the other matrix (or matrices). Similarly, to subtract one matrix from the other, subtract each element by the element in the other matrix in the corresponding position (as subtracting a matrix is the same as adding a negative matrix).

For example,

$$\begin{bmatrix} 3 & 9 \\ 7 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 16 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 8 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & -6 \end{bmatrix}$$

For addition and subtraction of matrices to be possible, the matrices must have the **same dimensions**. That is, they must have the same number of rows and columns.

Matrix Multiplication

Scalar Multiplication

Scalar multiplication involves multiplying a matrix by a constant. To do so, multiply each element within the matrix by the constant.

For example,

$$3 \times \begin{bmatrix} 8 & 4 & 2 \\ 4 & 0 & 2 \\ 1 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 6 \\ 12 & 0 & 6 \\ 3 & 18 & 15 \end{bmatrix}$$

Conformability

In order for two matrices to be multiplied, the number of rows in the first matrix needs to be equal to the number of columns in the second matrix. If they satisfy this criteria, then the matrix product is **conformable**.

In the following example, **AB** is conformable but **BA** is not,

$$\boldsymbol{A} = \begin{bmatrix} 8 & 4 \\ 4 & 0 \\ 1 & 6 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 8 & 6 \end{bmatrix}$$

Matrix Multiplication

To multiply matrices, you are required to find the **dot product** of rows and columns.

Finding the dot product involves multiplying each row of the first matrix with its corresponding column in the second matrix and then summing the multiples up, in order to arrive at a **single value** which will form an element in the resulting matrix.

The resulting element's position is determined by the row and column numbers of its multiples. That is, row one multiplied by column one will result in the element a_{11} in the resulting matrix. Necessarily then, the dimensions of the resulting matrix will reflect the number of rows in the first matrix and the number of columns in the second.

For example,

$$\begin{bmatrix} 3 & 0 & 1 \\ 5 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ 27 & 9 \end{bmatrix}$$

To obtain element $\boldsymbol{a_{11}}$ in the resulting matrix, the **first row** of the first matrix $(3 \quad 0 \quad 1)$ is multiplied by the **first column** of the second matrix $(2 \quad 1 \quad 5)$ such that $3 \times 2 \times + 0 \times 1 + 1 \times 5 = 11$.

The same process is used to find the remaining elements, for example, element $\boldsymbol{a_{12}}$ is found by multiplying the first row of the first matrix and the second column of the second matrix. And so on for the remaining elements.

Conformable Matrices

In order to multiply two matrices, the number of **rows of the first matrix** must equal the number of **columns of the second matrix.** That is, the matrices need to be conformable. Thus, the order matters in matrix multiplication and **AB** is not the same as **BA.**

Non-conformable Matrices

When attempting to multiply two matrices that are not conformable, the result will be undefined.

For example,

$$\begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \end{bmatrix} = undefined$$

Alternative Notation

Alternative notation of matrix multiplication is as follows:

$$AB = \left\{ \sum_{j=1}^{n} a_{ij} b_{jk} \right\}$$

The above notation precisely specifies the multiplication of two matrices and may be useful when more compact specification is required.

Vector Multiplication

Vector multiplication is just a special case of matrix multiplication where a row vector (i.e. a matrix with only 1 row) is multiplied by a column vector (i.e. a matrix with only one column). The same rules of matrix multiplication as specified above apply.

For example,

$$\begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [9]$$

In all vector multiplications, the resulting “matrix” is a single element.

Transpose of a Matrix

Transposing a matrix involves swapping the rows and columns such that the transpose of a matrix will have the **opposite** row and column dimensions.

In the example below, a 2×3 matrix becomes a 3×2 matrix.

$$\begin{bmatrix} 3 & 0 & 1 \\ 5 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

A **symmetric matrix** is a square matrix that is identical to its transpose. For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

A **diagonal matrix** is a symmetric matrix whose elements off the diagonal are all 0.

For example,

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Finding the Determinant

In order to find the inverse of a matrix, you will need first need to find its discriminant. The determinant can only be found for **square matrices** and the following sections illustrate how to find the discriminant of 2×2 and 3×3 matrices.

The determinant of a matrix $\boldsymbol{A}$ is generally denoted as $|\boldsymbol{A}|$

Discriminant of a 2 × 2

Given a matrix with two rows and two columns,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is calculated by: $|A| = ad - bc$

For example,

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

$$|A| = 2 \times 3 - 1 \times 5$$

$$|A| = 1$$

Discriminant of a 3 × 3

Given a matrix with three rows and three columns,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is calculated by: $|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$
For example,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 3 & 6 & 4 \end{bmatrix}$$

$$|A| = 2(3 \times 4 - 5 \times 2) - 0(0 \times 4 - 2 \times 3) + 1(0 \times 6 - 3 \times 3)$$

$$|A| = 2(2) - 0 + 1(9)$$

$$|A| = 13$$

Finding the Inverse

The inverse of a matrix is conceptually the same as the reciprocal of a number, such that if you multiply a matrix by its inverse, the result will be an **identity matrix**.

If a matrix has a non-zero determinant, then that matrix will have a unique inverse. Hence, while only square matrices can have inverses, not all square matrices have an inverse.

The inverse of a matrix $\boldsymbol{A}$ is generally denoted as $\boldsymbol{A^{-1}}$

To calculate the inverse of a 2×2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For example,

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Step 1: Find the determinant:

$$|A| = 2 \times 3 - 1 \times 5$$

$$|A| = 1$$

Step 2: Find the inverse

$$A^1 = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

To check that you have found the right answer, you can multiply the inverse with the original matrix. If the resulting matrix is an identity matrix, then you have obtained the correct inverse.

To check the example above,

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$