THE UNIVERSITY OF SYDNEY

Business School

Probability

Mathematics Help Sheet

The University of Sydney Business School

Introduction

What is probability?

Probability is the expression of knowledge or belief about the chance of something occurring. For example, it helps us answer questions such as:

- What are your chances of winning a raffle in which 500 people have bought 1 ticket each?
- If two dice are tossed, is it more likely that you will get a "3" and a "4" thrown or a "1" and a "1"?
- What are the chances that the bus will be late this morning?

When we express the probability of an event, the probability value will range from 0 to 1, that is, $0 \leq P(E) \leq 1$.

- A probability of 0 means the event is impossible
- A probability of 1 means the event will certainly occur
- A probability between 0 and 1 reflects the uncertainty of the event occurring

Notation and terminology

P(E)	The probability value of some event, "E", occurring.
Experiment	A process that produces a single outcome whose result cannot be predicted with certainty
Sample space	The collection of all outcomes that can result from a selection, decision, or experiment
Event	A subset of the sample space, representing an individual occurrence

Three concepts of probability

1. Classical probability

This type of probability is used to analyse situations where each outcome is equally possible, and is measured by taking the ratio of the number of ways a particular outcome can occur, to the number of ways all outcomes can occur. Expressed mathematically, it is:

$$\frac{Favourable\ outcomes}{All\ possible\ outcomes}$$

For example, when rolling a die there are 6 possible outcomes, and thus the chance of rolling a 6 would be 1/6.

2. Empirical probability

This probability is used when the total number of possible outcomes is unknown, and is calculated by conducting trials or experiments to represent a proxy of the entire population. Expressed mathematically, it is:

$$\frac{Observed\ favourable\ outcomes}{Total\ observed\ outcomes}$$

For example, to calculate the probability of a random person preferring apples over oranges, you could survey a sample of 1,000 people. If you found that 400 preferred apples, then the probability of a random person preferring apples would be 400/1000, or 40%.

3. Subjective probability

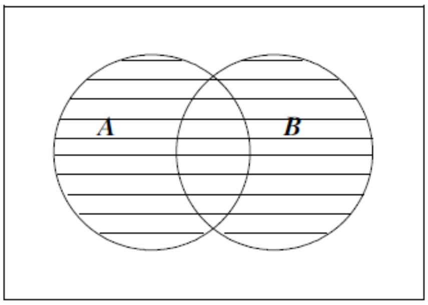
This probability is used when the number of outcomes is unknown and not measurable, and is simply a subjective estimate. Obviously this type of probability is the least accurate, however, in absence of better information on which to rely, subjective probability may be used to make logically consistent decisions. For example, your stock broker tells you he is 70% sure this stock will outperform next year.

Set notation and Venn diagrams

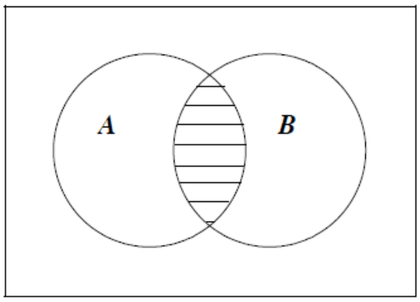
Set notation, which describes Venn diagrams, is frequently used in the context of probability to illustrate different scenarios and the mathematical formulas used under each one to calculate probability.

Set A collection of objects, described by listing its members, or elements, within parentheses {}. For example, A = {2, 4, 6, 8} means that A is the set consisting of numbers 2, 4, 6, and 8.

Union The union of A and B is the set of elements which belong to either A or B. Within Venn diagrams, this represents the area inside A and B. Using Set notation, this can be denoted by $A \cup B$.



Intersection The intersection of A and B is the set of elements which belong to both A and B, and is also denoted by $A \cap B$. This is illustrated on the diagram below.



Complement The complement of A, denoted by A', is the set of all elements which do not belong to A. In making this definition, we assume that all other elements belong to some larger set, "U", which is typically called the universal set, or the sample space. In the context of probability, when there are only two events, A and B, then P(B) = P(A')

Empty set The set with no elements in it, and can be denoted as {}.

Subset A set of objects that can be found within a larger set. For example, C = {2,4} is a subset of B = {2,4,6,8}.

Probability under different scenarios

The precise mathematical formula or method to calculate the probability of certain events will depend upon the circumstances in which the events occur. This section will provide an overview of the common scenarios that you may face.

Single event with finite number of equally likely outcomes

This scenario describes cases where there are a finite number of outcomes, each of which are equally likely to occur. For example, when you toss a coin, you know there are only two outcomes (heads or tails), and each outcome is as likely as the other to occur.

The probability of an event, "A", occurring under this scenario is:

$$P(A) = \frac{Number\ of\ elements\ in\ A}{Total\ number\ of\ elements\ in\ the\ sample\ space}$$

For example,

- If a coin is tossed, the probability of obtaining a head is $\frac{1}{2}$
- If a card is selected at random from a pack of 52 cards, the probability of obtaining a heart is $\frac{13}{52}$ or $\frac{1}{4}$

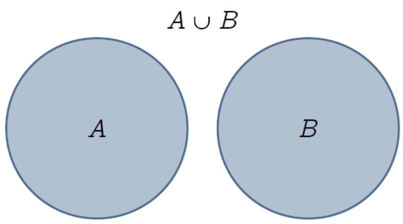
Multiple events – “either” / “or”

When we want to consider the probability of either one **OR** another event occurring (e.g. the probability of “A” or “B” occurring), we are essentially calculating the probability of the **union of A and B** (i.e. $A \cup B$). In calculating this, we will be concerned with whether the events are mutually exclusive, that is, whether they can occur at the same time.

1. Mutually exclusive events

This first scenario considers when the events within the sample space cannot occur at the same time, making them mutually exclusive. For example, when a die is rolled, it can only land on one face at a time, and thus they are mutually exclusive.

Considering an example where there are only two events, “A” and “B”, we can use a Venn diagram to depict this scenario. There is no overlap between the two circles as they are mutually exclusive.



Using Set notation, the mathematical formula for calculating the probability of **union of sets A and B** is given by,

$$P(A \cup B) = P(A) + P(B)$$

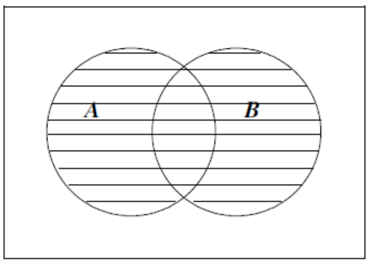
For example, to calculate the probability of rolling either a 1 or 6 when the die is rolled once,

$$P(1) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

2. Non-mutually exclusive events

This scenario considers when events cannot occur at the same time, thus making them non-mutually exclusive. For example, when calculating the probability that a randomly selected person is a female or is born in August, that person can be both a female and be born in August.

Again, we can use a Venn diagram to depict this scenario with the two circles now overlapping, showing that they are non-mutually exclusive.



Using Set notation, the probability of the union of "A" and "B" will be,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: $P(A \cap B)$ is the probability of event A **and** B occurring, which will be discussed in the next section.

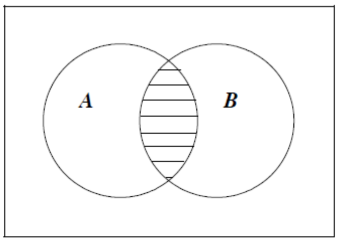
Multiple events – "and"

In this section, we will calculate the probability of two events occurring (e.g. "A" **and** "B" occurring). In calculating this, we will be concerned with whether the events are independent, that is, whether the event will occur, irrespective of whether or not another event has occurred.

1. Independent events

When the chance of a given outcome remains the same, irrespective of whether or not another event has occurred, the events are **independent**. For example, if a coin is tossed three times, the chance of the second toss landing on a head is not influenced by whatever the first toss landed on (i.e. the chance is still 0.5).

Using a Venn diagram, this is depicted as the intersection between "A" and "B".



The probability of independent events occurring is found by **multiplying their individual probabilities,**

$$P(A \cap B) = P(A) \times P(B)$$

For example, the probability of obtaining "6" and "6" on two successive die rolls is given by,

$$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

2. Non-independent events

When the chance of one event occurring is dependent on another event occurring, these events are not independent, and the probability is calculated differently. Refer to the conditional probability section, which is relevant for how to calculate the probability of two non-independent events occurring.

Conditional probability (Bayes' Theorem)

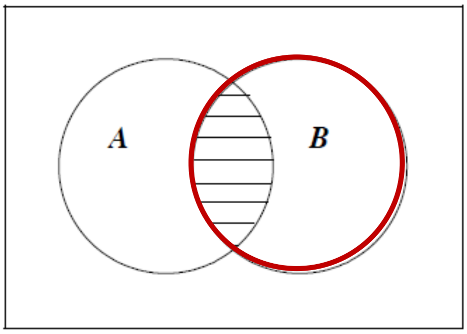
This scenario depicts cases where the probability of the event you are measuring is conditional upon another event occurring first. For example,

- What is the probability of A given B?
- What is the probability of being a girl given you were born in 1996?
- What is the probability of completing university given you have already started?

The **notation** we use to depict "probability of A given B" is

$$P(A|B)$$

Using a Venn diagram, we can frame conditional probability as shrinking the sample space to represent the conditionality of the event. For example, the "probability of A given B" would have its sample space reduced to the area of B, and the event is the intersection between A and B, as indicated below.



To calculate the conditional probability, we can use **Bayes' Theorem,**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

That is, the probability of event A, given event B, is calculated by the intersection of A and B divided by the probability of B occurring.

Permutations and Combinations

A permutation of a set of objects is an arrangement of objects in a certain order, hence when dealing with a permutation the order is important. For example, the number of possibilities of a lock combination is actually a permutation, since the order matters.

Where there is **no replacement** of objects, the number of permutations of a set of n objects taken r at a time is given by,

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Where n is the total number of objects, and r is the number of objects taken or chosen each time.

Where there is **replacement** of objects (that is objects can be repeated), the number of permutations of n objects taken r at a time, with repetition, is simply,

$$n^{r}$$

A combination of a set of objects is an unordered arrangement, that is, the order does not matter. For example, the selection of people in a team is a combination since generally who you select first does not matter.

The number of combinations of n objects taken r at a time is given by,

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!\,(n-r)}$$

Where n is the total number of objects (e.g. total number of people to choose from), and r is the number of objects taken or chosen each time (e.g. the number of people on the team).