Mathematics Learning Centre

The University of Sydney

The composite function rule (the chain rule)

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1 The composite function rule (also known as the chain rule)

Have a look at the function $f(x) = (x^2 + 1)^{17}$. We can think of this function as being the result of combining two functions. If $g(x) = x^2 + 1$ and $h(t) = t^{17}$ then the result of substituting $g(x)$ into the function h is

$$h(g(x)) = (g(x))^{17} = (x^2 + 1)^{17}.$$

Another way of representing this would be with a diagram like

$$x \overset{g}{\longmapsto} x^2 + 1 \overset{h}{\longmapsto} (x^2 + 1)^{17}.$$

We start off with x. The function g takes x to $x^2 + 1$, and the function h then takes $x^2 + 1$ to $(x^2 + 1)^{17}$. Combining two (or more) functions like this is called *composing* the functions, and the resulting function is called a *composite function.* For a more detailed discussion of composite functions you might wish to refer to the Mathematics Learning Centre booklet *Functions*.

Using the rules that we have introduced so far, the only way to differentiate the function $f(x) = (x^2 + 1)^{17}$ would involve expanding the expression and then differentiating. If the function was $(x^2 + 1)^2 = (x^2 + 1)(x^2 + 1)$ then it would not take too long to expand these two sets of brackets. But to expand the seventeen sets of brackets involved in the function $f(x) = (x^2 + 1)^{17}$ (or even to expand using the binomial theorem) would take a long time. The composite function rule shows us a quicker way.

Rule 7 (The composite function rule (also known as the chain rule))

If $f(x) = h(g(x))$ then $f'(x) = h'(g(x)) \times g'(x)$.

In words: differentiate the 'outside' function, and then multiply by the derivative of the 'inside' function.

To apply this to $f(x) = (x^2 + 1)^{17}$, the outside function is $h(\cdot) = (\cdot)^{17}$ and its derivative is $17(\cdot)^{16}$. The inside function is $g(x) = x^2 + 1$ which has derivative $2x$. The composite function rule tells us that $f'(x) = 17(x^2 + 1)^{16} \times 2x$.

As another example let us differentiate the function $1/(z^3 + 4z^2 - 3z - 3)^6$. This can be rewritten as $(z^3 + 4z^2 - 3z - 3)^{-6}$. The outside function is $(\cdot)^{-6}$ which has derivative $-6(\cdot)^{-7}$. The inside function is $z^3 + 4z^2 - 3z - 3$ with derivative $3z^2 + 8z - 3$. The chain rule says that

$$\frac{d}{dz}(z^3 + 4z^2 - 3z - 3)^{-6} = -6(z^3 + 4z^2 - 3z - 3)^{-7} \times (3z^2 + 8z - 3).$$

There is another way of writing down, and hence remembering, the composite function rule.

Rule 7 (The composite function rule (alternative formulation))

If y is a function of u and u is a function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This makes the rule very easy to remember. The expressions $\frac{dy}{du}$ and $\frac{du}{dx}$ are not really fractions but rather they stand for the derivative of a function with respect to a variable. However for the purposes of remembering the chain rule we can think of them as fractions, so that the du cancels from the top and the bottom, leaving just $\frac{dy}{dx}$.

To use this formulation of the rule in the examples above, to differentiate $y = (x^2 + 1)^{17}$ put $u = x^2 + 1$, so that $y = u^{17}$. The alternative formulation of the chain rules says that

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 17u^{16} \times 2x \\ &= 17(x^2 + 1)^{16} \times 2x.\end{aligned}$$

which is the same result as before. Again, if $y = (z^3 + 4z^2 - 3z - 3)^{-6}$

then set $u = z^3 + 4z^2 - 3z - 3$ so that $y = u^{-6}$ and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -6u^{-7} \times (3z^2 + 8z - 3).\end{aligned}$$

You select the formulation of the chain rule that you find easiest to use. They are equivalent.

Example

Differentiate $(3x^2 - 5)^3$.

Solution

The first step is always to **recognise** that we are dealing with a composite function and then to split up the composite function into its components. In this case the outside function is $(\cdot)^3$ which has derivative $3(\cdot)^2$, and the inside function is $3x^2 - 5$ which has derivative $6x$, and so by the composite function rule,

$$\frac{d(3x^2 - 5)^3}{dx} = 3(3x^2 - 5)^2 \times 6x = 18x(3x^2 - 5)^2.$$

Alternatively we could first let $u = 3x^2 - 5$ and then $y = u^3$. So

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 6x = 18x(3x^2 - 5)^2.$$

Example

Find $\frac{dy}{dx}$ if $y = \sqrt{x^2+1}$.

Solution

The outside function is $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ which has derivative $\frac{1}{2}(\cdot)^{-\frac{1}{2}}$, and the inside function is x^2+1 so that

$$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x.$$

Alternatively, if $u = x^2+1$, we have $y = \sqrt{u} = u^{\frac{1}{2}}$. So

$$\frac{dy}{dx} = \frac{1}{2}\,u^{-\frac{1}{2}} \times 2x = \frac{1}{2}\,(x^2+1)^{-\frac{1}{2}} \times 2x.$$

Exercise 1.1

Differentiate the following functions using the composite function rule.

a. $(2x+3)^2$ **b.** $(x^2+2x+1)^{12}$ **c.** $(3-x)^{21}$

d. $(x^3-1)^5$ **e.** $f(t) = \sqrt{t^2-5t+7}$ **f.** $g(z) = \frac{1}{\sqrt{2-z^4}}$

g. $y = (t^3 - \sqrt{t})^{-3.8}$ **h.** $z = (x + \frac{1}{x})^{\frac{3}{7}}$

Exercise 1.2

Differentiate the functions below. You will need to use both the composite function rule and the product or quotient rule.

a. $(x+2)(x+3)^2$ **b.** $(2x-1)^2(x+3)^3$ **c.** $x\sqrt{(1-x)}$

d. $x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$ **e.** $\dfrac{x}{\sqrt{1-x^2}}$

Solutions to exercises

Exercise 1.1

a. $\dfrac{d}{dx}\Big((2x+3)^2\Big) = 8x+12$

b. $\dfrac{d}{dx}\Big((x^2+2x+1)^{12}\Big) = 12(x^2+2x+1)^{11}(2x+2)$

c. $\dfrac{d}{dx}\Big((3-x)^{21}\Big) = -21(3-x)^{20}$

d. $\dfrac{d}{dx}\Big((x^3-1)^5\Big) = 5(x^3-1)^4 3x^2 = 15x^2(x^3-1)^4$

e. $\dfrac{d}{dt}\sqrt{t^2-5t+7} = \dfrac{d}{dt}(t^2-5t+7)^{\frac{1}{2}} = \dfrac{1}{2}(t^2-5t+7)^{-\frac{1}{2}}(2t-5)$

f. $\dfrac{d}{dz}\left(\dfrac{1}{\sqrt{2-z^4}}\right) = \dfrac{d}{dz}\Big((2-z^4)^{-\frac{1}{2}}\Big) = 2z^3(2-z^4)^{-\frac{3}{2}}$

g. $\dfrac{d}{dt}\Big((t^3-\sqrt{t})^{-3.8}\Big) = -3.8(t^3-\sqrt{t})^{-4.8}(3t^2-\dfrac{1}{2\sqrt{t}})$

h. $\dfrac{d}{dx}\Big((x+\dfrac{1}{x})^{\frac{3}{7}}\Big) = \dfrac{3}{7}(x+\dfrac{1}{x})^{-\frac{4}{7}}(1-\dfrac{1}{x^2})$

Exercise 1.2

a. $\dfrac{d}{dx}\Big((x+2)(x+3)^2\Big) = (x+3)^2 + 2(x+2)(x+3)$

b. $\dfrac{d}{dx}\Big((2x-1)^2(x+3)^3\Big) = 4(2x-1)(x+3)^3 + 3(2x-1)^2(x+3)^2$

c. $\dfrac{d}{dx}\Big(x\sqrt{1-x}\Big) = \sqrt{1-x} - \dfrac{x}{2\sqrt{1-x}}$

d. $\dfrac{d}{dx}\Big(x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}\Big) = \dfrac{1}{3}x^{-\frac{2}{3}}(1-x)^{\frac{2}{3}} - \dfrac{2}{3}x^{\frac{1}{3}}(1-x)^{-\frac{1}{3}}$

e. $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \dfrac{\sqrt{1-x^2}+x^2(1-x^2)^{-\frac{1}{2}}}{1-x^2}$