Elementary row operations

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Systems of linear equations

The system of linear equations

$$\begin{aligned} 3x + 4y - z &= 1 \\ x - y + z &= 3 \\ -x + 2y - 3z &= 5 \end{aligned}$$

can be written in matrix form as:

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

The matrix equation of a system of linear equations

In the matrix equation $\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$ is the matrix of the coefficients of x, y, and z,

and $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a column of constants from the right hand sides of the equations.

If we multiply the matrices on the left hand side we can read off our system of linear equations again.

The augmented matrix

All the information in the matrix equation

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

can be summarised into one matrix called the **augmented matrix**

$$\left[\begin{array}{ccc|c} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right].$$

Here the coefficient matrix has been augmented with the column of constants.

Elementary row operations

There are three elementary row operations that we can perform on an augmented matrix.

The elementary row operations are:

1. interchanging two rows
2. multiplying a row by a non-zero constant
3. adding a multiple of one row to another

The elementary row operations are reversible and so do not change the solutions of our original system of linear equations.

Interchanging two rows

Consider the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right].$$

We can interchange the first and second rows as follows:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 3 & 4 & -1 & 1 \\ -1 & 2 & 3 & 5 \end{array}\right] \text{ indicated by } R_1 \longleftrightarrow R_2.$$

This row operation is easily reversed by $R_1 \longleftrightarrow R_2$ again.
This means that the solution set of the system of linear equations is unchanged.

Multiplying a row by a non-zero constant

Consider the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right].$$

We can multiply row three by a non-zero constant of -2:

$$\left[\begin{array}{rrr|r} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & -4 & -6 & -10 \end{array}\right] \text{ indicated by } R_3 \longrightarrow (-2)R_3.$$

Again this row operation can be reversed by $R_3 \longrightarrow (-\frac{1}{2})R_3$ so the solution set remains unchanged.

Adding a multiple of one row to another

Consider the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right].$$

We can add $2\times$ row 2 to row 1:

$$\left[\begin{array}{rrr|r} 5 & 2 & 1 & 7 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right] \text{ indicated by } R_1 \longrightarrow R_1 + 2R_2.$$

This row operation is reversible by $R_1 \longrightarrow R_1 + (-2)R_2$, so the solution set remains unchanged.