Mathematics Learning Centre

The University of Sydney

Exponents

Christopher Thomas

1 Exponents

1.1 Introduction

Whenever we use expressions like 7^3 or 2^5 we are using exponents.

The symbol 2^5 means $\underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ factors}}$. This symbol is spoken as 'two raised to the power five', 'two to the power five' or simply 'two to the five'. The expression 2^5 is just a shorthand way of writing 'multiply 2 by itself 5 times'. The number 2 is called the *base*, and 5 the *exponent*.

Similarly, if b is any real number then b^3 stands for $b \times b \times b$. Here b is the base, and 3 the exponent.

If n is a whole number, b^n stands for $\underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}$. We say that b^n is written in *exponential form*, and we call b the base and n the exponent, power or index.

Special names are used when the exponent is 2 or 3. The expression b^2 is usually spoken as 'b squared', and the expression b^3 as 'b cubed'. Thus 'two cubed' means $2^3 = 2 \times 2 \times 2 = 8$.

1.2 Exponents with the same base

We will begin with a very simple definition. If b is any real number and n is a positive integer then b^n means b multiplied by itself n times. The rules for the behaviour of exponents follow naturally from this definition.

Rule 1: $b^n \times b^m = b^{n+m}$.

That is, to *multiply* two numbers in exponential form (with the same base), we *add* their exponents.

Rule 2: $\frac{b^n}{b^m} = b^{n-m}$.

In words, to *divide* two numbers in exponential form (with the same base) , we *subtract* their exponents.

We have not yet given any meaning to negative exponents, so n must be greater than m for this rule to make sense. In a moment we will see what happens if n is not greater than m.

Rule 3: $(b^m)^n = b^{mn}$

That is, to raise a number in exponential form to a power, we *multiply* the exponents.

Until now we have only considered exponents which are positive integers, such as 7 or 189. Our intention is to extend this notation to cover exponents which are not necessarily positive integers, for example -5, or $\frac{113}{31}$, or numbers such as $\pi \approx 3.14159$.

Also, we have not attached any meaning to the expression b^0. It doesn't make sense to talk about a number being multiplied by itself 0 times. However, if we want rule 2 to continue to be valid when $n = m$ then we must *define* the expression b^0 to mean the number 1.

If $b \neq 0$ then we define b^0 to be equal to 1. We do not attempt to give any meaning to the expression 0^0. It remains undefined.

We initially had no idea of how to extend our notation to cover a zero exponent, but if we wish rules 1, 2 and 3 to remain valid for such an exponent then the definition $b^0 = 1$ is forced on us. We have no choice.

We have come up with a sensible definition of b^0 by taking $m = n$ in rule 2 and seeing what b^0 must be if rule 2 is to remain valid. To come up with a suitable meaning for negative exponents we can take $n < m$ in rule 2. For example, let's try $n = 2$ and $m = 3$.

Rule 2 gives

$$\begin{aligned} \frac{b^2}{b^3} &= b^{-1} \quad \text{or} \\ \frac{1}{b} &= b^{-1}. \end{aligned}$$

This suggests that we should define b^{-1} to be equal to $\frac{1}{b}$. This definition, too, makes sense for all values of b except $b = 0$.

In a similar way we can see that we should define b^{-n} to mean $\frac{1}{b^n}$, except when $b = 0$, in which case it is undefined. You should convince yourself of this by showing that the requirement that rule 2 remains valid forces on us the definitions

$$\begin{aligned} b^{-2} &= \frac{1}{b^2} \quad \text{and} \\ b^{-3} &= \frac{1}{b^3}. \end{aligned}$$

If n is a positive integer (for example $n = 17$ or $n = 178$) then we define b^{-n} to be equal to $\frac{1}{b^n}$. This definition makes sense for all values of b except $b = 0$, in which case the expression b^{-n} remains undefined.

Pause for a moment and look at what has been achieved. We have been able to give a meaning to b^n for all integer values of n, positive, negative, and zero, and we have done it in such a way that all three of the rules above still hold. We can give meaning to expressions like $(\frac{35}{7})^{13}$ and π^{-7}.

We have come quite a way, but there are a lot of exponents that we cannot yet handle. For example, what meaning would we give to an expression like $5^{\frac{7}{9}}$? Our next task is to give a suitable meaning to expressions involving fractional powers.

Let us start with $b^{\frac{1}{2}}$. If rule 2 is to hold we must have

$$b^{\frac{1}{2}} \times b^{\frac{1}{2}} = b^{\frac{1}{2}+\frac{1}{2}} = b^1 = b.$$

So, $b^{\frac{1}{2}}$ is defined to be the positive square root of b, also written $\sqrt{b}$. So $b^{\frac{1}{2}} = \sqrt{b}$.

Of course, b must be positive if $b^{\frac{1}{2}}$ is to have any meaning for us, because if we take any real number and multiply itself by itself then we get a positive number. (Actually there is a way of giving meaning to the square root of a negative number. This leads to the notion of complex numbers, a beautiful area of mathematics which is beyond the scope of this booklet.)

That takes care of a meaning for $b^{\frac{1}{2}}$ if $b > 0$. Now have a look at $b^{\frac{1}{3}}$. If rule 2 is to remain valid then we must have

$$b^{\frac{1}{3}} \times b^{\frac{1}{3}} \times b^{\frac{1}{3}} = b^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = b^1 = b.$$

In general if we wish we wish to give meaning to expressions like $b^{\frac{1}{n}}$ in such a way that rule 3 holds then we must have $(b^{\frac{1}{n}})^n = b^1 = b$.

If b is positive, $b^{\frac{1}{n}}$ is defined to be a positive number, the n^{th} root of b. That is, a number whose n^{th} power is equal to b. This number is sometimes written $\sqrt[n]{b}$.

If b is negative we need to look at separately at the cases where n is even and where n is odd.

If n is *even* and b is negative, $b^{\frac{1}{n}}$ cannot be defined, because raising any number to an even power results in a positive number.

If n is *odd* and b is negative, $b^{\frac{1}{n}}$ can be defined. It is a negative number, the n^{th} root of b. For example, $(-27)^{\frac{1}{3}} = -3$ because $(-3) \times (-3) \times (-3) = -27$.

Now we can see how to define $b^{\frac{p}{q}}$ for any number of the form $\frac{p}{q}$, where p and q are integers. Such numbers are called *rational numbers*.

Notice that $\frac{p}{q} = p \times \frac{1}{q}$, so if rule 3 is to hold then $b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}$.

We know how to make sense of $(b^{\frac{1}{q}})^p$ and $(b^p)^{\frac{1}{q}}$, and they turn out to be equal, so this tells us how to make sense of $b^{\frac{p}{q}}$. If we want rules 1, 2 and 3 to hold then we must define $b^{\frac{p}{q}}$ to be either one of $(b^p)^{\frac{1}{q}}$ or $(b^{\frac{1}{q}})^p$.

This definition always makes sense when b is positive, but we must take care when b is negative. If q is even then we may have trouble in making sense of $b^{\frac{p}{q}}$ for negative b. For example we cannot make sense of $(-3)^{\frac{3}{2}}$. This is because we cannot even make sense of $(-3)^{\frac{1}{2}}$, let alone $((-3)^{\frac{1}{2}})^3$. Trying to take the exponents in the other order does not help us because $(-3)^3 = -27$ and we cannot make sense of $(-27)^{\frac{1}{2}}$.

However it may be that the numerator and denominator of $\frac{p}{q}$ contain common factors which, when cancelled, leave the denominator odd. For example we can make sense of $(-3)^{\frac{4}{6}}$, even though 6 is even, because $\frac{4}{6} = \frac{2}{3}$, and we can make sense of $(-3)^{\frac{2}{3}}$. A rational number $\frac{p}{q}$ is said to be expressed in its *lowest form* if p and q contain no common factors. If $\frac{p}{q}$, when expressed in its lowest form, has q odd then we can make sense of $b^{\frac{p}{q}}$ even for $b < 0$.

To recapitulate, we define

$$b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}.$$

This definition makes sense for all $\frac{p}{q}$ if $b > 0$. If $b < 0$ then this definition makes sense providing that $\frac{p}{q}$ is expressed in its lowest form and q is odd.

So far, if $b > 0$, we have been able to give a suitable meaning to b^x for all rational numbers x. Not every number is a rational number. For example, $\sqrt{2}$ is an *irrational* number: there do not exist integers p and q such that $\sqrt{2} = \frac{p}{q}$. However for $b > 0$ it is possible to extend the definition of b^x to irrational exponents x so that rules 1, 2 and 3 remain valid. Thus if $b > 0$ then b^x is defined for all real numbers x and satisfies rules 1, 2 and 3. We will not show how b^x may be defined for irrational numbers x.

Examples

$(\frac{1}{3})^{-1} = \frac{1}{(\frac{1}{3})} = 3$

$(0.2)^{-3} = \frac{1}{(0.2)^3} = \frac{1}{0.008} = 125$

$(-64)^{\frac{2}{3}} = [(-64)^{\frac{1}{3}}]^2 = (-4)^2 = 16$ or,

$(-64)^{\frac{2}{3}} = [(-64)^2]^{\frac{1}{3}} = (4096)^{\frac{1}{3}} = 16$

$16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8$

$(-16)^{\frac{3}{4}}$ is not defined.

$5^{\frac{3}{2}} = 5^{1+\frac{1}{2}} = 5 \times 5^{\frac{1}{2}} = 5\sqrt{5}$

1.3 Exponents with different bases

From the definition of exponents we know that if n is a positive integer then

$$\begin{aligned}
(ab)^n &= \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{n \text{ factors}} \\
&= \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}} \times \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}} \qquad \text{(switching the order around)} \\
&= a^n b^n.
\end{aligned}$$

Just as in section 1.2, we can show that this equation holds true for more general exponents than integers, and we can formulate the following rule:

Rule 4: $(ab)^x = a^x b^x$ whenever both sides of this equation make sense, that is, when each of $(ab)^x$, a^x and b^x make sense.

Again, from the definition of exponents we know that if n is a positive integer then

$$\begin{aligned}
\left(\frac{a}{b}\right)^n &= \underbrace{\frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b}}_{n \text{ factors}} \qquad (b \neq 0) \\
&= \frac{\overbrace{a \times a \times \cdots \times a}^{n \text{ factors}}}{\underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}} \\
&= \frac{a^n}{b^n}
\end{aligned}$$

As in section 1.2, we can show that this equation remains valid if the integer n is replaced by a more general exponent x. We can formulate the following rule:

Rule 5: $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$ whenever both sides of this equation make sense, that is, whenever $(\frac{a}{b})^x$, a^x and b^x make sense.

An expression of the form $a^x b^y$ cannot generally be simplified, though it can be written in the form $(ab^{\frac{y}{x}})^x$ or $(a^{\frac{x}{y}}b)^y$ if necessary. For example, we cannot really make the expression a^2b^5 any simpler than it is, though we could write it in the form $(ab^{\frac{5}{2}})^2$ or $(a^{\frac{2}{5}}b)^5$.

Examples

$(2 \times 3)^3 = 2^3 \times 3^3 = 8 \times 27 = 216 = 6^3$

$(4x)^{\frac{1}{2}} = 4^{\frac{1}{2}}x^{\frac{1}{2}} = 2x^{\frac{1}{2}} = 2\sqrt{x}$

$(-40)^{\frac{1}{3}} = (-8 \times 5)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} \times (5)^{\frac{1}{3}} = -2 \times \sqrt[3]{5}$

$(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$

$(\frac{4}{7})^{-2} = \frac{1}{(\frac{4}{7})^2} = 1 \times \frac{7^2}{4^2} = \frac{49}{16}$

$(-\frac{27}{8})^{-\frac{1}{3}} = (-\frac{8}{27})^{\frac{1}{3}} = \frac{(-8)^{\frac{1}{3}}}{27^{\frac{1}{3}}} = -\frac{2}{3}$

1.4 Summary

If $b > 0$ then b^x is defined for all numbers x. If $b < 0$ then b^x is defined for all integers and all numbers of the form $\frac{p}{q}$ where p and q are integers, $\frac{p}{q}$ is expressed in its lowest form and q is odd. The number b is called the *base* and x is called the *power, index* or *exponent.* Exponents have the following properties:

1. If n is a positive integer and b is any real number then $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}$.

2. $b^{\frac{1}{n}} = \sqrt[n]{b}$, and if n is even we take this to mean the positive n^{th} root of b.

3. If $b \neq 0$ then $b^0 = 1$. b^0 is undefined for $b = 0$.

4. If p and q are integers then $b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}$.

5. $b^x \times b^y = b^{x+y}$ whenever both sides of this equation are defined.

6. $\dfrac{b^x}{b^y} = b^{x-y}$ whenever both sides of this equation are defined.

7. $b^{-x} = \dfrac{1}{b^x}$ whenever both sides of this equation are defined.

8. $(ab)^x = a^x b^x$ whenever both sides of this equation are defined.

9. $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$ whenever both sides of this equation are defined.

1.5 Exercises

The following expressions evaluate to quite a 'simple' number. If you leave some of your answers in fractional form you won't need a calculator.

1. $9^{\frac{1}{2}}$ **2.** $16^{\frac{3}{4}}$ **3.** $(\frac{1}{5})^{-1}$ **4.** $(3^{-1})^2$ **5.** $(\frac{5}{2})^{-2}$

6. $(-8)^{\frac{3}{2}}$ **7.** $(\frac{-27}{8})^{\frac{2}{3}}$ **8.** $5^{27}5^{-24}$ **9.** $8^{\frac{1}{2}}2^{\frac{1}{2}}$ **10.** $(-125)^{\frac{2}{3}}$

These look a little complicated but are equivalent to simpler ones. 'Simplify' them. Again, you won't need a calculator.

11. $\dfrac{3^{n+2}}{3^{n-2}}$ **12.** $\sqrt{\dfrac{16}{x^6}}$ **13.** $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$

14. $(x^2 + y^2)^{\frac{1}{2}} - x^2(x^2 + y^2)^{-\frac{1}{2}}$ **15.** $\dfrac{x^{\frac{1}{2}} + x}{x^{\frac{1}{2}}}$ **16.** $(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}})$

1.6 Solutions to exercises

1. $9^{\frac{1}{2}} = \sqrt{9} = 3$

2. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$

3. $(\frac{1}{5})^{-1} = \frac{1}{\frac{1}{5}} = 5$

4. $(3^{-1})^2 = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

5. $(\frac{5}{2})^{-2} = (\frac{2}{5})^2 = \frac{4}{25}$

6. $(-8)^{\frac{3}{2}}$ is not defined.

7. $(\frac{-27}{8})^{\frac{2}{3}} = ((\frac{-27}{8})^{\frac{1}{3}})^2 = (\frac{-3}{2})^2 = \frac{9}{4}$

8. $5^{27}5^{-24} = 5^{27-24} = 5^3 = 125$

9. $8^{\frac{1}{2}}2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$

10. $(-125)^{\frac{2}{3}} = ((-125)^{\frac{1}{3}})^2 = (-5)^2 = 25$

11. $\frac{3^{n+2}}{3^{n-2}} = 3^{n+2-(n-2)} = 3^4 = 81$

12. $\sqrt{(\frac{16}{x^6})} = (\frac{16}{x^6})^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{x^{6\times\frac{1}{2}}} = \frac{4}{x^3}$

13. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2 + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2 = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$

14.

$$\begin{aligned}
(x^2+y^2)^{\frac{1}{2}} - x^2(x^2+y^2)^{-\frac{1}{2}} &= (x^2+y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{(x^2+y^2)^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}} - x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{x^2+y^2-x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{y^2}{(x^2+y^2)^{\frac{1}{2}}}
\end{aligned}$$

15. $\frac{x^{\frac{1}{2}}+x}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} = 1 + x^{\frac{1}{2}}$

16.

$$\begin{aligned}
(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}}) &= u^{\frac{1}{3}}u^{\frac{2}{3}} + u^{\frac{1}{3}}(uv)^{\frac{1}{3}} + u^{\frac{1}{3}}v^{\frac{2}{3}} - v^{\frac{1}{3}}u^{\frac{2}{3}} - v^{\frac{1}{3}}(uv)^{\frac{1}{3}} - v^{\frac{1}{3}}v^{\frac{2}{3}} \\
&= u - v
\end{aligned}$$