Mathematics Learning Centre

The University of Sydney

The first derivative and stationary points

Jackie Nicholas

The first derivative and stationary points

The derivative $\dfrac{dy}{dx}$ of a function $y = f(x)$ tell us a lot about the shape of a curve. In this section we will discuss the concepts of stationary points and increasing and decreasing functions.

However, we will limit our discussion to functions $y = f(x)$ which are well behaved. Certain functions cause technical difficulties so we will concentrate on those that don't!

The first derivative

The derivative, $\frac{dy}{dx}$, is the slope of the tangent to the curve $y = f(x)$ at the point x. If we know about the derivative we can deduce a lot about the curve itself.

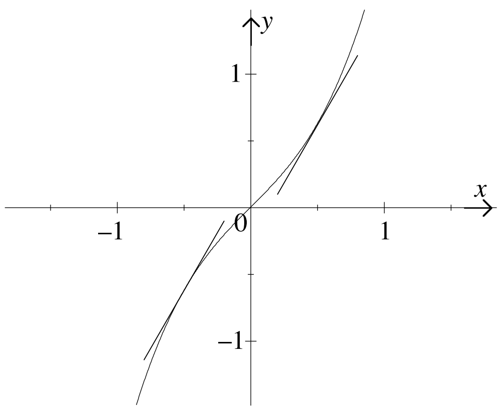
Increasing functions

If $\dfrac{dy}{dx} > 0$ for all values of x in an interval I, then we know that the slope of the tangent to the curve is positive for all values of x in I and so the function $y = f(x)$ is *increasing* on the interval I.

For example, let $y = x^3 + x$, then

$$\frac{dy}{dx} = 3x^2 + 1 > 0 \quad \text{for all values of } x.$$

That is, the slope of the tangent to the curve is positive for all values of x. So, $y = x^3 + x$ is an increasing function for all values of x.



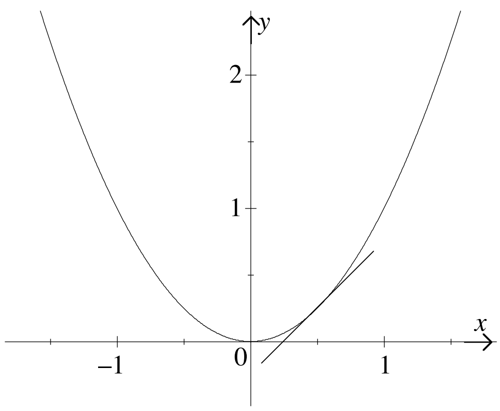
The graph of $y = x^3 + x$.

We know that the function $y = x^2$ is increasing for $x > 0$. We can work this out from the derivative.

If $y = x^2$ then

$$\frac{dy}{dx} = 2x > 0 \quad \text{for all } x > 0.$$

That is, the slope of the tangent to the curve is positive for all values of $x > 0$. So, $y = x^2$ is increasing for all $x > 0$.



The graph of $y = x^2$.

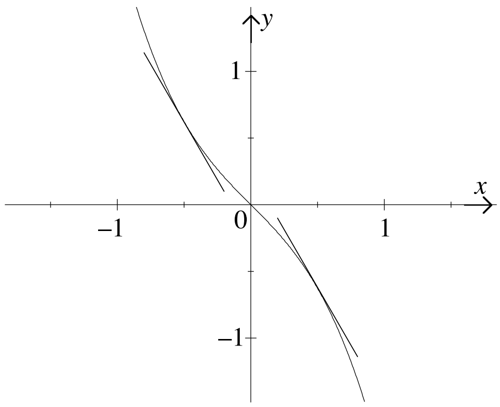
Decreasing functions

If $\dfrac{dy}{dx} < 0$ for all values of x in an interval I, then we know that the slope of the tangent to the curve is negative for all values of x in I and so the function $y = f(x)$ is *decreasing* on the interval I.

For example, let $y = -x^3 - x$, then

$$\frac{dy}{dx} = -3x^2 - 1 < 0 \quad \text{for all values of } x.$$

That is, the slope of the tangent to the curve is negative for all values of x. So, $y = -x^3 - x$ is a decreasing function for all values of x.



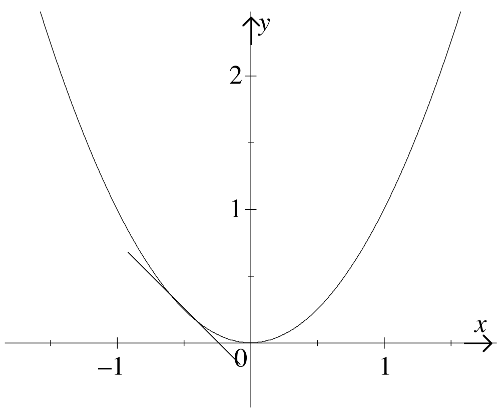
The graph of $y = -x^3 - x$.

We know that the function $y = x^2$ is decreasing for $x < 0$. We can work this out from the derivative.

If $y = x^2$ then

$$\frac{dy}{dx} = 2x < 0 \quad \text{for all } x < 0.$$

That is, the slope of the tangent to the curve is negative for all values of $x < 0$. So, $y = x^2$ is decreasing for all $x < 0$.



The graph of $y = x^2$.

Stationary points

When $\dfrac{dy}{dx} = 0$, the slope of the tangent to the curve is zero and thus horizontal. The curve is said to have a *stationary point* at a point where $\dfrac{dy}{dx} = 0$.

There are three types of stationary points. They are relative or local maxima, relative or local minima and horizontal points of inflection. Relative or local maxima and minima are so called to indicate that they may be maxima or minima only in their locality. They are also called turning points. Points of inflection are defined and discussed later.

The nature of stationary points

The first derivative can be used to determine the nature of the stationary points once we have found the solutions to $\frac{dy}{dx} = 0$.

Relative maximum

Consider the function $y = -x^2 + 1$. By differentiating and setting the derivative equal to zero, $\frac{dy}{dx} = -2x = 0$ when $x = 0$, we know there is a stationary point when $x = 0$.
We can use the fact that this is the only stationary point (and that we are only dealing with well behaved functions) to divide the real line into two intervals; $x < 0$ and $x > 0$.
From the derivative we know that since

$$\frac{dy}{dx} = -2x > 0 \qquad \text{when } x < 0$$

the function is increasing for $x < 0$. (Choose a point which is < 0, say $x = -1$, as a test point.)
Similarly, since

$$\frac{dy}{dx} = -2x < 0 \qquad \text{when } x > 0$$

the function is decreasing for $x > 0$. (We could use the point $x = 1$ as a test point here.)
Of course, at the point $x = 0$ itself $\frac{dy}{dx} = 0$.
Putting all this together, we can deduce that the stationary point at $x = 0$ is a relative maximum.

Since

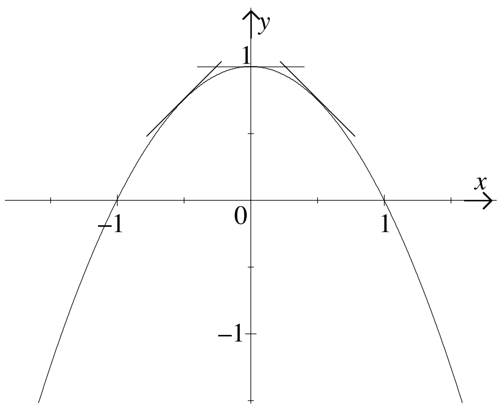
$$\frac{dy}{dx} > 0 \quad \text{for } x < 0,$$

$$\frac{dy}{dx} = 0 \quad \text{for } x = 0,$$

and

$$\frac{dy}{dx} < 0 \quad \text{for } x > 0,$$

we have a relative maximum at $x = 0$.



The graph of $y = -x^2 + 1$.

We can sum this information up in a table that allows us to see at a glance that the point $(0, 1)$ is a maximum.

x	< 0	0	> 0
y'	$+ve$	0	$-ve$
y	$\nearrow$	1	$\searrow$

Relative minimum

Consider the function $y = x^2 - 2x + 3$. By differentiating and setting the derivative equal to zero, $\dfrac{dy}{dx} = 2x - 2 = 0$ when $x = 1$, we know there is a stationary point at $x = 1$.

Again, we use the fact that this is the only stationary point to divide the real line into two intervals; $x < 1$ and $x > 1$.

From the derivative we know that since

$$\frac{dy}{dx} = 2x - 2 = 2(x - 1) < 0 \qquad \text{when } x < 1$$

the function is decreasing for $x < 1$. (Choose a point which is < 1, say $x = 0$, as a test point.)

Similarly, since

$$\frac{dy}{dx} = 2(x - 1) > 0 \qquad \text{when } x > 1$$

the function is increasing for $x > 1$. (We could use the point $x = 2$ as a test point here.)

Of course, at the point $x = 1$ itself $\frac{dy}{dx} = 0$.

Therefore, we deduce that the stationary point at $x = 1$ is a relative minimum.

Since

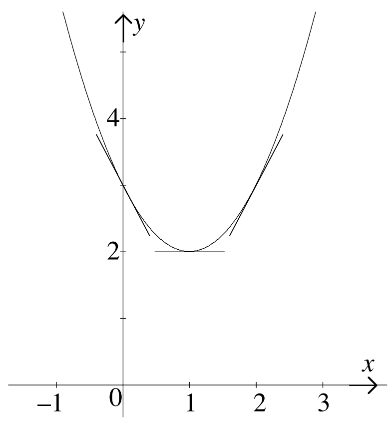
$$\frac{dy}{dx} < 0 \quad \text{for } x < 1,$$

$$\frac{dy}{dx} = 0 \quad \text{for } x = 1,$$

and

$$\frac{dy}{dx} > 0 \quad \text{for } x > 1,$$

we have a relative minimum at $x = 1$.



The graph of $y = x^2 - 2x + 3$.

Again we can summarise this in a table.

x	< 1	1	> 1
y'	$-ve$	0	$+ve$
y	$\searrow$	1	$\nearrow$

We analyse functions with more than one stationary point in the same way.

Example

Consider $y = 2x^3 - 3x^2 - 12x + 4$. Then,

$$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1).$$

So, $\dfrac{dy}{dx} = 0$ when $x = -1$ or $x = 2$.

Therefore the points $(-1, 11)$ and $(2, -16)$ are the only stationary points. As before, we use the stationary points to partition the real line into the following intervals: $x < -1$, $-1 < x < 2$, $x > 2$.

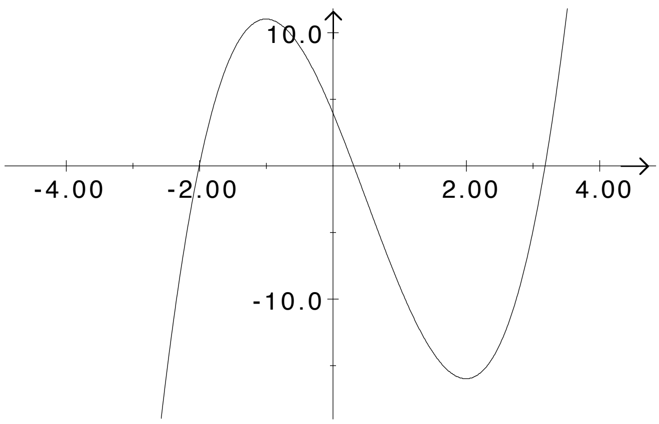
We can now choose test points in these intervals, say $x = -2$, $x = 0$ and $x = 3$, to determine the sign of the derivative in these intervals.

x	< -1	-1	$> -1, < 2$	2	> 2
y'	$+ve$	0	$-ve$	0	$+ve$
y	$\nearrow$	11	$\searrow$	-16	$\nearrow$

Therefore the point $(-1, 11)$ is a maximum and the point $(2, -16)$ is a minimum.

When $x = 0$, $y = 4$ so the point $(0, 4)$ lies on the graph of the function.

We now have enough information to sketch the graph.



The graph of $y = 2x^3 - 3x^2 - 12x + 4$.