Mathematics Learning Centre

The University of Sydney

Graphs of polynomials

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1 Graphs of Polynomials

Polynomials are functions like $y = 4x + 1$, $y = 7x^3 - 3x - 3$ and $y = 3x^4 - 2x$.

In general, a polynomial function has the general form

$$y = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where the coefficients a_0, a_1, a_2... a_n are constants, and n is any positive integer. The number n (the highest power that appears) is called the degree of the polynomial.

1.1 Some Familiar Polynomials

You can save yourself time and trouble by knowing something about what shape to expect. We have already seen that equations of the form

$$y = ax + b$$

are straight lines.

1.1.1 Quadratic Functions

A quadratic function is a function of the form

$$y = ax^2 + bx + c$$

where $a \neq 0$, and a, b, and c are constants. The graph of a quadratic is always a parabola and looks something like Figure 1 or Figure 2.

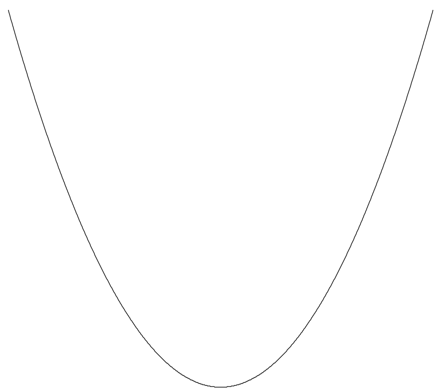

Fig. 1

or

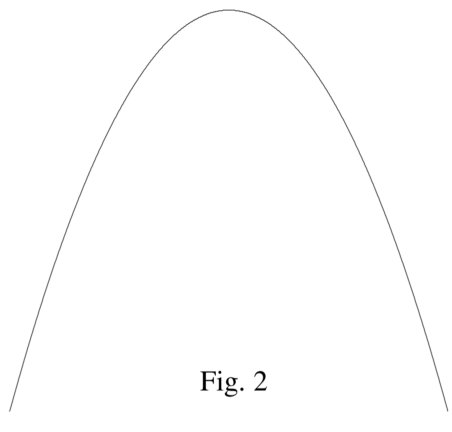

Fig. 2

1.1.2 Cubic Functions

A cubic is a function of the form

$$y = ax^3 + bx^2 + cx + d$$

where $a \neq 0$, and a, b, c and d are constants. If $a > 0$ the graph of a cubic looks similar to one of the graphs in Figures 3, 4, and 5.

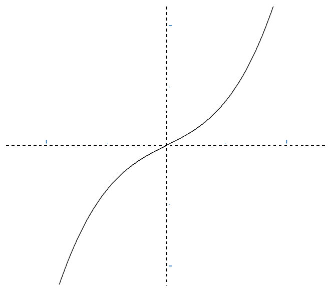

Fig. 3

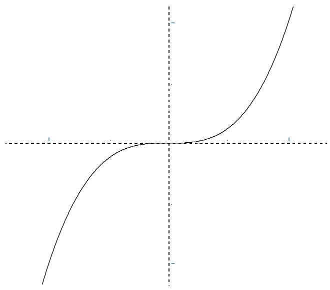

Fig. 4

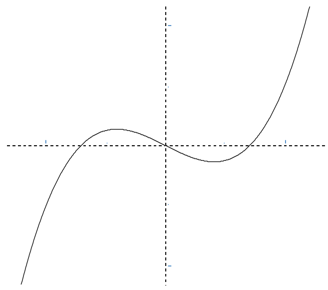

Fig. 5

If $a < 0$, then the above curves will be reflected in the X axis as illustrated in Figures 6, 7, and 8.

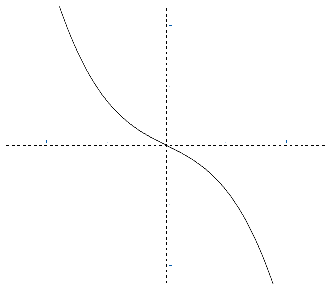

Fig. 6

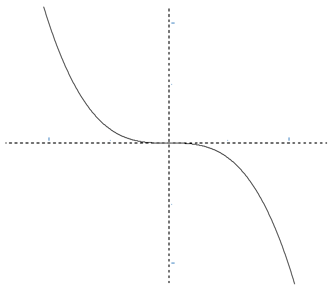

Fig. 7

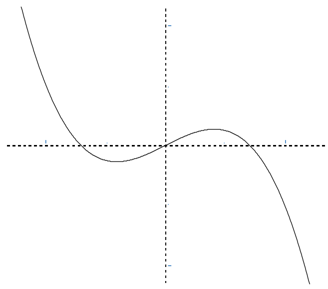

Fig. 8

Let us consider a function of the form $y = ax^3 + bx^2 + cx + d$, where a is positive. All we know at the moment is that its graph looks like one of Figures 3, 4 or 5. We know that when x is a large positive number y will also be large and positive, while if x is a large negative number y will also be large and negative. This is because the term ax^3 in the equation of the polynomial is positive if $x > 0$ and negative if $x < 0$, and this term dominates the others for large values of x.

We extend this idea for polynomials of higher degree in the next section.

1.2 General Points about Polynomials

1.2.1 The basic shape

A polynomial function is always defined and continuous for all real numbers. This means that there are no sudden jumps or holes in the curve. Also we will see in a later lecture that a polynomial can be differentiated for all values of x. The derivative is also a polynomial and so it too is continuous everywhere. This means that the graph of a polynomial is smooth everywhere–there are no sharp corners (called cusps) where it suddenly changes direction.

1.2.2 Behaviour when x is large positive and large negative

One of the important pieces of information that can help us to draw a graph is the way it behaves for large positive and large negative values of x. The term of the polynomial with the highest power of x is called the **dominant term**.

When x is very large positive or very large negative, this term determines how the graph behaves, because it is so much bigger in magnitude than all the other terms. How the graph behaves for x large in magnitude depends on the **power** and the **coefficient** of the dominant term. There are four possibilities which we summarise in the following diagrams. These show the direction of the curve for large positive and large negative values of x with reference to the dominant term.

Even power, positive coefficient.

e.g. $y = x^2$

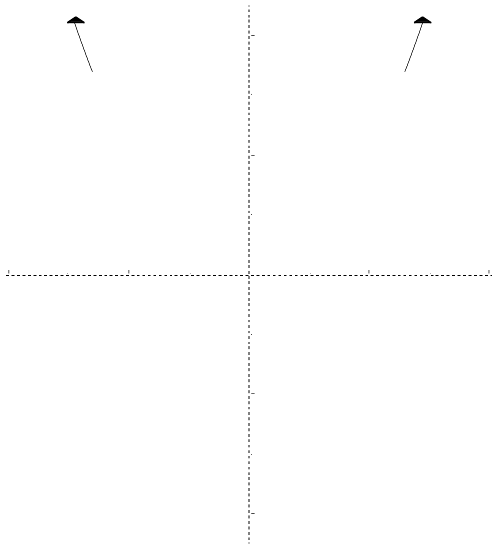

Fig. 9

Even power, negative coefficient.

e.g. $y = -x^2$

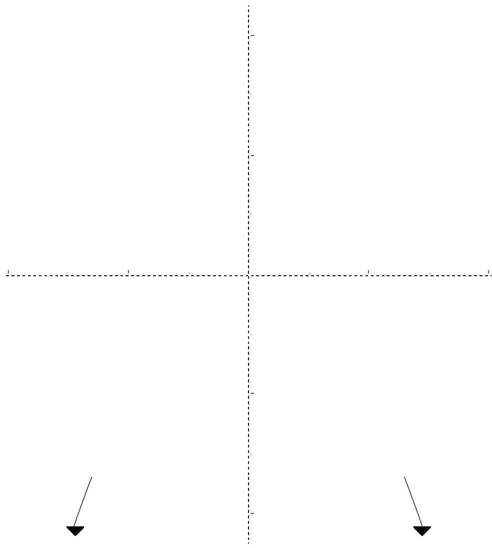

Fig. 10

Odd power, positive coefficient.

e.g. $y = x^3$

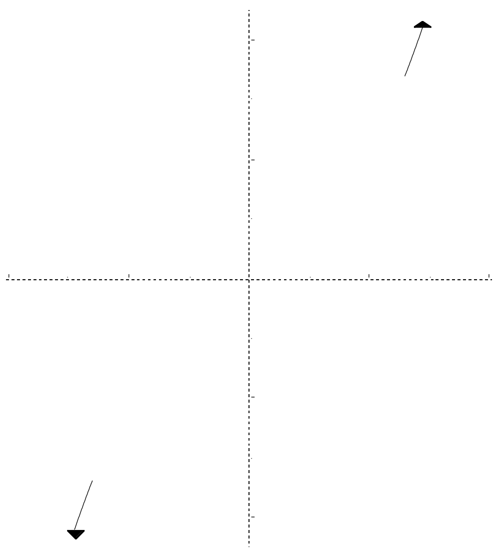

Fig. 11

Odd power, negative coefficient.

e.g. $y = -x^3$

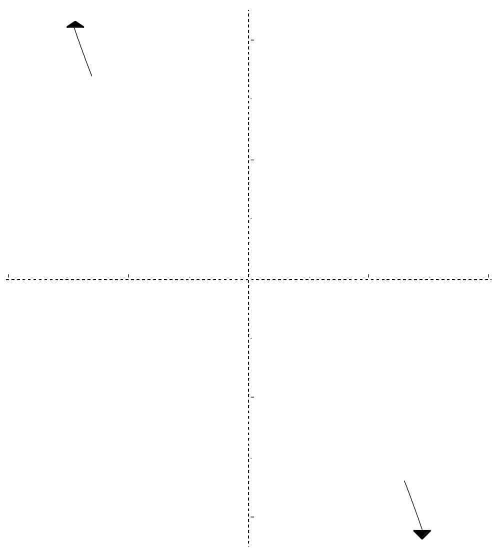

Fig. 12

Example: Discuss the behaviour of the function $y = -2x^5 + x^4 - 6x^2 - 10$, for values of x large in magnitude.

Solution: Since the greatest power of x is five, and its coefficient is negative, we have a function which behaves like that in Figure 12.

Example: Discuss the behaviour of the function $y = x^8 + 3$, for values of x large in magnitude.

Solution: Since the power of the dominant term is even and its coefficient positive, the function behaves like that in Figure 9 for large positive and large negative values of x.

This gives a good start to graphing the polynomial. All we need to do now is work out what happens in between.

To do this, we need to know where the curve increases and decreases, where there are stationary points (i.e. turning points and stationary points of inflection) and other points of inflection. We will use calculus to find these things out.