The inverse matrix

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The identity matrix

Recall, we defined an identity matrix as a square matrix with 1's down the main diagonal and 0's everywhere else.

So $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix,

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3×3 identity matrix, and

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the 4×4 identity matrix.

Identity matrices

Identity matrices are important as they have the property that when we multiply a matrix A by the appropriate identity matrix, the product is A itself.

If A is a 2×3 matrix, then

$$I_2 \times A = A = A \times I_3.$$

Example: If $A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}$ then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}.$$

Check that $A \times I_3 = A$ for yourself.

Inverse matrices

Let A be an $n \times n$ matrix.

The inverse of a matrix A is a matrix B such that

$$AB = BA = I_n.$$

If $AB = BA$ then matrix B must also be $n \times n$.

Thus only square matrices can have an inverse.

An inverse matrix is unique

If A is a square $n \times n$ matrix which has an inverse, the inverse is unique.

Notation

The inverse of a matrix A, if it exists, is denoted by A^{-1} so

$$AA^{-1} = A^{-1}A = I.$$

Note, not all square matrices have inverses.

The inverse of a 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

We define the *determinant* of A as

$$\det A = |A| = ad - bc.$$

The matrix A has an inverse A^{-1} if

$$\det A = |A| = ad - bc \neq 0,$$

in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Examples

Let $A = \begin{bmatrix} 0 & -5 \\ 1 & -4 \end{bmatrix}$, then $\det A = 0 - (-5)(1) = 5 \neq 0$

so the inverse A^{-1} exists.

$$A^{-1} = \frac{1}{5} \begin{bmatrix} -4 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & 1 \\ -\frac{1}{5} & 0 \end{bmatrix}.$$

Let $B = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$, then $\det B = 2(-3) - (-6)(1) = 0.$

So B is not invertible, ie B^{-1} does not exist.