Modelling and Calculus

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1.1 Rates of Change and Derivatives

1.1.1 Identifying The Difference Between a Quantity and the Rate of Change of That Quantity

Imagine a block of ice put in one of your classrooms.

The block of ice will melt.

The warmer the room the quicker the ice will melt.

How do we write this as an equation?

Let us say the volume of the block of ice is given by V.

Then the rate at which the block will melt will be $\frac{dV}{dt}$, where t is the time.

Remember the $\frac{d}{dt}$ represents how quickly something changes.

The rate of change of volume, in words, is represented by $\frac{dV}{dt}$ in mathematics.

Now the temperature of the room influences **how quickly** the ice will melt.

The temperature does not change the volume of ice instantaneously. A hot room doesn't mean there will be no ice straight away.

So the temperature of the room influence $\frac{dV}{dt}$ **not** V directly.

In nature and the physical world often one quantity will influence the rate of change of another.

Here we are making the distinction between a quantity and how quickly that quantity changes. These are not inter-changable concepts.

There may be lots of ice in a freezer. In which case the volume is large but the ice is changing very slowly.

There may be an iceberg which has drifted to a location with a warm climate in which case there is lots of ice changing very quickly.

There may be an ice cube in a cold slushy drink in which case there is little ice changing slowly.

There may be an ice cube in a warm beer in which case there is little ice that will change rapidly.

This distinction between the amount or quantity of a substance, and the rate of change of that substance is a critical concept for understanding modelling and calculus.

In written words expressions like **rate**, **rate of change**, **speed**, **acceleration**, **how quickly**, **how slowly**, **how fast**, amongst others may indicate a derivative in mathematics.

1.1.2 Exercises

In the following sentences identify which part, or parts, of the sentence represents $\dfrac{dx}{dt}$ or the derivative of a quantity.

For each question you need to make a distinction between which part of the question describes the quantity of something in the problem and which parts describe the rate of change of that quantity in the problem; just as we made a distinction between the volume of ice and the rate at which the volume of ice changes in the examples above.

1. A cold sausage is placed in an oven. The rate of increase of temperature of the sausage will depend on how hot the oven is.
2. The rate of change of concentration of salt in a cell will depend on the difference between the concentration of salt in the cell and in the concentration of salt in the environment.
3. The rate at which a human body produces insulin will depend on the concentration of sugars in the blood
4. How fast a car travels will depend on the rate at which fuel is being taken from the fuel tank and fed to the engine.
5. How fast a car accelerates or changes velocity depends on the rate of change of the rate at which fuel is being taken from the fuel tank and fed to the engine.
6. A swimming pool, which is initially full of water, is drained through a hole in the bottom of the pool. The rate at which the depth of water drops will depend on the the pressure at the bottom of the pool and hence will depend on the depth of water the water in the pool.
7. A ballon is filled with air and then allowed to deflate. The larger the balloon the more pressure the air will exert. The rate of change of the volume of the balloon will depend on the diameter of the balloon.

1.2 What is a Differential Equation?

1.2.1 Description of a Differential Equation

In nature, very often one property of a system will influence the rate of change of another property.

For instance if we place a hot pie in a warm room the rate of change of the temperature of the pie will depend on how cold or warm the room is. If the pie is put in a freezer the pie will cool quickly. If the pie is put in a warm room it will cool more slowly. If placed in an oven it will cool slowly or even heat up, depending on how hot the oven is, and in this case how hot the pie is.

The rate of change of the temperature of the pie will depend on the temperature of the room.

Many, many physical, chemical, electrical, biological and other natural systems can be well modeled by relationships between one property of the system and the rate of change of another.

Many, many physical, chemical, electrical, biological and other natural systems can be well modeled by relationships between one property of the system and the rate of change of another.

For these reasons we need to incorporate derivatives into our equations.

A differential equation is an equation which involves a derivative of one of the variables.

1.2.2 Constant Rate of Change

If we put a hose in a swimming pool and turn the tap on full the pool will fill up.

If the tap delivers water at, let's say, 1000 litres every hour and this rate doesn't change then the volume of water in the pool will change by a certain amount in any given time period.

If the volume of water in the pool is V then the rate of change of V will be constant.

In mathematics this idea is simply written as:

$$\frac{dV}{dt} = c,$$

where c is a constant and t is time.

This mathematical expression has an equal sign hence it is called an equation.

The equation also involved a derivative $\frac{dV}{dt}$ hence it is called a differential equation.

The expression

$$\frac{dV}{dt} = c,$$

is an equation and has a derivative of one of the variables; hence we call it a differential equation.

If a pool initially has 2000 litres of water in it and then has 3000 litres of water after constantly filling for 1 hour and 4000 litres of water after 2 hours and 6000 litres after 4 hours then

$$\frac{dV}{dt} = 1000\ \frac{\text{litres}}{\text{hour}},$$

or, for V measured in litres and t measured in hours we just write

$$\frac{dV}{dt} = 1000.$$

Here the change in volume for any fixed period of time is the same. The volume changes by 1000 in any hour. The rate of change in volume will depend on how much we turn the tap on. If we turn-off the tap a bit the rate will decrease, but the volume will not.

The volume of water in the pool is different than the rate of increase of volume.
The rate at which the water enters the pool may be 1000 litres per hour, but the volume in the pool is never 1000 litres.

1.2.3 Exercises

For the following worded questions identify which parts of the question represents the derivative of a quantity, which part represents the quantity itself, which part represents the equality, and which feature of the problems tell us what the rate of change will be.

1. A ballon fills with air such that the rate of change of volume of the balloon is equal to 2000 litres per minute.
2. The rate of change of temperature of a cold beer is equal to the difference in temperature of the beer and the room.
3. The rate at which a swimming pool is filled is equal to a constant times the opening in the tap.
4. A bowling ball is thrown out of an aircraft and falls such that the rate of change of the speed is increasing by 9.8 metres per second every second. (This means that the object, which initially is not falling at all will be falling at 9.8 metres per second after 1 second, and will be falling at 19.6 metres per second after 2 seconds etcetera.)
5. A balloon deflates such that the rate of change of volume of the balloon is equal to a constant times the volume of the balloon. In this case as the balloon deflates there is less pressure exerted by the balloon on the air inside.
6. The rate of change of concentration of alcohol in the blood stream of a person who has stopped drinking is constant.
7. The change of speed of an object falling to Earth will increase at a constant rate of 9.8 metres per second every second. (This means that if the object is dropped from rest then it will be falling at a speed of 9.8 metres per second after one second and then be falling at a speed of 19.6 metres per second after 2 seconds.)
8. The rate of change of the wind-speed (where the wind-speed is positive if it is onshore and negative if is or offshore) in an idealized costal region will be dependent on the difference in the temperature of the land and the ocean.
9. The rate at which a small shark population increases will equal a constant times the number of fish in their habitat

1.3 Proportionality

1.3.1 Description of Proportional Quantities

The diameter of a circle is always two times the radius.

For a circle of radius 1 metre the diameter is 2.

For a circle of radius 1 light year the diameter is 2 light years. A light year is how far light travels in a year (in a vacuum). This is a pretty big circle.

Likewise there is a relationship between the distance around a circle (the circumference) and the diameter.

For a tractor wheel of diameter of 1 metre the circumference is about $3.141\ldots$ metres $= \pi$ m.
For a ferris wheel of diameter of 10 metres the circumference is about $31.41\ldots$ metres $= 10\pi$ m.

> If a fixed change in one quantity always leads to, a not necessarily equal, but fixed change in another we say that the two quantities are proportional.

For instance if we change the radius of a circle by 1 metre then we will change the diameter by 2 metres. If we increase the diameter of a circle by 1 metre then we will change the circumference by $3.141\ldots$ m or π m.

We say that the diameter is proportional to the radius or in mathematical symbols we write

$$\text{diameter} \propto \text{radius} \qquad \text{or} \qquad d \propto r,$$

where d is the diameter of the circle and r is the radius.

This means that the diameter will be equal to a constant times the radius. Here

$$d = 2r.$$

Likewise the circumference is proportional to the diameter of a circle or

$$\text{circumference} \propto \text{diameter} \qquad \text{or} \qquad c \propto d,$$

where c is the circumference. Here the circumference is equal to a constant times the diameter, and the constant is the most famous constant of proportionality, π. We write

$$c = \pi d$$

> If two quantities A and B are proportional we write
>
> $$A \propto B,$$
>
> which reads A is proportional to B.

> If A and B are proportional then A will be equal to a non-zero constant times B or
>
> $$A = k \times B,$$
>
> which reads A is equal to a non zero constant k times B.

1.3.2 Exercises

Are the following quantities proportional?

1. The diameter of a sphere and the radius of a sphere.
2. The circumference of a circle and the diameter of a circle.
3. The circumference of a great circle (the largest circumference of a sphere) and the diameter of a sphere.
4. The perimeter of a square and the length of a side of a square.
5. The area of a square and the side length of a square.
6. The area of a circle and the radius of a circle.
7. The height and the corresponding weight of each person in a group of 100 students.
8. The number of people passing a maths class and the size of that maths.
9. The area of a circle and the radius-squared of the circle.
10. The volume of a cube and the volume of the biggest sphere that can just be contained in that sphere.

1.3.3 Constants of Proportionality

If A is proportional to B then

$$A = k \times B,$$

where $k \neq 0$ is called the constant of proportionality.

It is generally specified that the constant of proportionality not be equal to zero. If a constant c could be zero then $A = c \times B$ would mean that $A = 0$ and A would be identically equal to zero, no matter what the value of B. For this circumstance the value of A would not be dependent on B or the two quantities A and B would not be dependent. Since we reasonably expect two quantities that are proportional to be dependent we generally exclude the case of $k = 0$ from the definition.

Perhaps the most famous constant of proportionality is the constant π, which is about $3.141\ldots$.

We know the diameter d and circumference of a circle c are proportional, or $c \propto d$ in this case

$$c = \pi d\,.$$

So π is the constant of proportionality between the diameter and circumference of a circle.

In fact this can be used as a definition of the important constant π.

π is the geometric constant of proportionality (for Euclidean space).

For a fixed change of say 1 metre in diameter, the circumference will change by $3.141\ldots$ or π metres.

If we increase the diameter of a Ferris wheel from 10 metres to 11 meters it will be $3.141\ldots$ metres larger around the circumference.

In fact if we change the diameter of any Ferris wheel by 1 metre, no matter how big or small, the circumference will change by π metres.

This is precisely why, when you pump up the tyres on a car or a bike the speedo will read slower when you are going at exactly the same speed along the road. Or another way of saying this is that for the same reading on the speedo “more pumped up” tyres will make the car travel faster.

If we graph the diameter of a ferris wheel with the circumference the graph will be a straight line. Every time we increase d by one metre c will increase by π metres.

The graph of any two proportional quantities with A on the horizontal axis B on the vertical axis, with say $A = k \times B$ will be a straight line with gradient k. If we change A by ΔA this will lead to a change in B of $k \times \Delta B$

1.3.4 Exercises

For each of the following sets of quantities find the constant of proportionality

1. The radius of a sphere and the diameter of a sphere.
2. The diameter of a sphere and the circumference of a great circle (the largest circumference of a sphere).
3. The length of a side of a square and the distance around the perimeter of a square.
4. The radius of a circle and the circumference of a circle.
5. The volume of a cube and the side length cubed of the cube.
6. The volume of a sphere and the radius of a sphere cubed.
7. An inch and a millimetre.
8. A litre and a cubic metre.
9. The side length of a square and the diagonal of a square.
10. The area of a square and the area of the largest circle that can wholly be contained in that square.
11. The volume of a cube and the volume of the biggest sphere that can wholly be contained in that cube. Hint: for d half the side length of the cube and r the radius of the sphere then $r^2 = d^2 + 2d^2$.
12. What constant of proportionality would you choose for the following quantity. The number of people passing a maths class and the size of that maths class.
13. What constant of proportionality would you choose for the following quantity. The number of people failing a maths class and the size of that maths class.
14. The area of a circle and the radius squared of the circle.

Chapter 2

Modelling and Calculus 2 MAC 2 Understanding the concepts and ideas of differential equations and their solutions in terms of written word descriptions of the differential equations and the concepts of solving a differential equation.

2.1 Differential Equations as Questions: Various Variables

2.1.1 Solution Function and Independent Variable

To work out what question a differential equation is asking we look at the symbol at the top of the derivative, for $\dfrac{dy}{dx}$ this is y. For $\dfrac{df}{dt}$ this is f. We may want to eventually find this function. Let's call this the solution function.

We then look at the thing we are differentiating with respect to, or the symbol on the bottom of the derivative. For $\dfrac{dy}{dx}$ this is x. For $\dfrac{df}{dt}$ this is t. Here we will call this the independent variable, because (here) we want to find the solution function in terms of this independent variable.

Here we want to find the solution function in terms of the independent variable.

For $\dfrac{dy}{dx}$ we want to find y in terms of x.
For $\dfrac{df}{dt}$ we want to find f in terms of t.

2.1.2 Independent Variables and Dependent Variables

The reason that we call one variable independent and one variable a dependent variable is that they will serve different rolls in the solution.

For instance if we find y in terms of x, let's say for instance we find $y = x^2 + 3x$ then we can change the x variable and simply calculate the new value for y. y is dependent on x and we think of x, as being able to be changed independently.

For the same function $y = x^2 + 3x$ if we change y it is quite an involved process to find the new value of x. Changing y results in a more involved calculation to find x.

Even though, strictly speaking, both variable are dependent on the other, through our equation; since we can't change one without the other, if we change the independent variable we can simply use our formula to find the new value of the dependent variable.

In some instances we may need to find the variable on the bottom of the derivative in terms of the variable on the top. In this case then the roles of the variables will be reversed. Here, to start with, we stick with the solution function on the top and the independent variable on the bottom of the derivative.

2.1.3 Alternative Notations for Differential Equations

There are many different notations for a derivative. We can write the derivative $\dfrac{dy}{dx}$ as $y'(x)$.

So for a derivative of the form $X'(t)$ for instance, we want to find the solution function X in terms of the independent variable t.

For this alternative notation:

For $Y'(x)$ we want to find Y in terms of x.
For $X'(t)$ we want to find X in terms of t.

For a differential equation of the form $\dfrac{dy}{dx} = 3x^2$, we want to find a solution function y as a function of x.

For a differential equation of the form $X'(t) = 20X$, we want to find a solution function X as a function of t.

Example 1

For the differential equation
$$\frac{dy}{dx} = 4x^2 + 2x,$$
identify the solution function and the independent variable.

Solution
Here y is the solution function and x is the independent variable. We want to find y as a function of x.

Example 2

For the differential equation
$$f'(z) = 2z^2,$$
identify the solution function and the independent variable.

Solution
Here f is the solution function and z is the independent variable. We want to find f as a function of z.

Example 3

For the differential equation
$$X'(t) = 15X,$$
identify the solution function and the independent variable.

Solution
Here X is the solution function and t is the independent variable. We want to find X as a function of t.

2.1.4 Exercises

For the following differential equations find the solution function(s) and the independent variable.
Note: you do not need to solve the differential equations here just name the solution function and the independent variable.

1. $\frac{dy}{dx} = x$

2. $\frac{dy}{dx} = y$

3. $\frac{dy}{dx} = 3x^2 + 5y^2$

4. $\frac{dy}{dx} = x$

5. $\frac{dx}{dy} = y^2 + 2x \times y$

6. $\frac{dx}{dt} \times x + t = 0$

7. $\frac{dx}{dt} = 3x^2 + 4t$

8. $0 = \frac{dz}{dy} + 2y \times z$

9. $3 + 7\frac{dz}{dx} + \left[\frac{dz}{dx}\right]^2 = 0$

10. $\frac{1}{\frac{dy}{dx}} + 3y = 0$

11. $\frac{dx}{dt} + y = t, \quad \frac{dy}{dt} + x = t^2$

12. $X'(t) = 4$

13. $X'(t) = X(t)$

14. $X'(t) + 3X(t) = t^2$

15. $X'(Z) + Z \times X = 0$

16. $Y'(Z) + 3 = Y(Z)$

17. $X'(t) - Xt = X^3t^2$

18. $X'(t) = X - Y, \quad Y'(t) = Y - X + 2$

19. $\frac{d^2X}{dt^2} + \frac{dX}{dt} + X$

20. $X''(t) + X'(t) + X(t) = 0$

2.2 Differential Equations as Questions

2.2.1 Differential Equations of the Form $\dfrac{dy}{dx} = f(x)$ as Questions

With the solution function and independent variable in mind we look at the rest of the differential equation.

It is useful to think of a differential equation, not as a mathematical formula, but as asking a question in words.

Example 1

The differential equation

$$\frac{dy}{dx} = 2x$$

is like asking the question:

> Can you think of a function y, which is a function of x, such that when you differentiate that function you get $2x$?

Just one answer to this question is

$$y = x^2,$$

or $y(x) = x^2$. Since if you differentiate x^2 you get $2x$.

We could have also chosen $y(x) = x^2 + 1$ for instance.

Example 2

The differential equation

$$\frac{dy}{dx} = 3x^2$$

is asking:

> Can you think of a function y, which is a function of x, such that when you differentiate that function you get $3x^2$?

Just one answer to that question is

$$y = x^3.$$

Can you think of another?

Example 3

The differential equation

$$\frac{dy}{dx} = 4x^3 + 10x^3$$

is asking:

> Can you think of a function y, which is a function of x, such that when you differentiate that function you get $4x^3 + 10x^3$?

2.2.2 Exercises

For the following differential equations write down a simple sentence in words that represents a question that the differential equation is asking.

1. $\dfrac{dy}{dx} = 2$
2. $\dfrac{dy}{dx} = 2x$
3. $\dfrac{dy}{dx} = \sin(x)$
4. $\dfrac{dy}{dx} = 3x^2$
5. $\dfrac{dy}{dx} = e^x$
6. $\dfrac{dy}{dx} = x\sin(x)$
7. $\dfrac{dy}{dx} + 2x = 3$
8. $X'(t) = 3t^2$
9. $Y'(z) = \sin z + z$
10. $W'(t) + \sin t = 15$
11. $\dfrac{dy}{dx} \times x = 3$
12. $\dfrac{dy}{dx} \times \sin(x) = x\sin(x)$
13. $\dfrac{dy}{dx} \times e^x = e^{2x}$

    Below are some harder questions that can be answered by extending the ideas in this section.

14. $\left[\dfrac{dy}{dx}\right]^2 = (3x^2 + 2)^2$
15. $X'(t) + X(t) = 3t^2$
16. $[X'(t)]^2 + X'(t) = 13$
17. $\left(\dfrac{dy}{dx}\right)^2 \times y + \sin(x) = 0$
18. $Z'(Y) \times Z(Y) + 3Z(Y) = Y^2$
19. $2\dfrac{dy}{dx} + 3y + \sin(x) = 0$
20. $[X'(t)]^2 + X(t)\sin t = t^2$

2.2.3 Differential Equations of the Form $\dfrac{dy}{dx} = g(y)$ as Questions

In all of our examples so far the right hand side has only involved independent variables. For the example above there are only xs on the right.

But there can be solution functions on the right as well.

Example 1

For instance we can have a differential equation such as

$$\frac{dy}{dx} = y.$$

This differential equation is, in essence, quite different to the others we have discussed above. The equation can be thought of as asking the question:

> Can you think of a function y, which is a function of x, such that when you differentiate that function you get the same function that you started with?

The function $y = e^x$ has derivative

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x = y.$$

So $y = e^x$ is a function that has itself as its derivative.

So $y = e^x$ is **just one** answer to our question—what function is its own derivative.

But again this is not the only function that answers our question.

The function $y = 2e^x$ has derivative

$$\frac{dy}{dx} = \frac{d}{dx}2e^x = 2e^x = y.$$

So $y = 2e^x$ is also a function that is its own derivative.

So $y = 2e^x$ is again **just one** answer to our question.

The functions $y = e^x$ and $y = 2e^x$ are answers to our question and there are many more. Other answers to the question "can you think of a function that is its own derivative" are $y = 3e^x$, $y = 24e^x$, $y = -351e^x$.

Indeed the exponential function is unique as it is the only type of function, such that when you differentiate it you get back the same function.

The only class of functions that are their own derivatives are the exponential functions of the form $y(x) = Ae^x$. Since

$$\frac{dy}{dx} = \frac{d}{dx}Ae^x = Ae^x = y$$

that is when we differentiate Ae^x we get back the same function we started with.

Hence Ae^x is the most general answer to our question.

The exponential function is the only type of function who's derivative is the same as the function itself.

$y(x) = \text{constant} \times e^x$ is the only type of function that is its own derivative.

Example 2

The differential equation

$$\frac{dy}{dx} = 4y$$

is more complicated again. It is asking:

> Can you think of a function y, which is a function of x, such that when you differentiate that function, you get 4 times the function that you first thought of?

Just one function which answers our question is $y = e^{4x}$, since here $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{4x} = 4 \times e^{4x}$ which is 4 times the function we first thought of. Can you think of others (Example 1 provides a hint)?

Example 3

The differential equation

$$\frac{dx}{dt} = x^2 + 2x$$

is asking:

> Can you think of a function x, which is a function of t, such that when you differentiate that function you get the square of the function you first thought of plus 2 times the function that you first thought of?

The function which answers our question is $x(t) = \dfrac{2}{ce^{-2t} + 1}$.

As we can see the solutions to differential equations get very complicated very quickly. In fact we can write down very simple looking differential equations that do not have solutions in terms of simple functions. One such example is $f'(x) = e^{x^2}$. There are **many** more.

In fact writing down differential equation can be thought of as a way of defining many special functions. The differential equation $\dfrac{dy}{dx} = y$ can be used as a definition for the function $y(x) = e^x$ and can be used to find the important constant $e = 2.718281828\ldots$.

2.2.4 Differential Equations of the Form $\dfrac{dy}{dx} = f(x)g(y)$ as Questions

Example 4

The differential equation

$$\frac{dx}{dt} = 2tx$$

is asking:

> Can you think of a function x, which is a function of t, such that when you differentiate that function you get $2t$ times the function that you first thought of?

Just one function which answers our question is $x = e^{t^2}$, since here $\dfrac{dx}{dt} = \dfrac{d}{dt}e^{t^2} = 2t \times e^{t^2}$ which is $2t$ times the function we first thought of.

Please note that you are not expected to solve these differential equations here. You should understand how to understand what a differential equation may be asking.

2.2.5 Exercises

1. Which question is the following differential equation asking in words?

$$\frac{dy}{dx} = 3x^2$$

(a) Can you think of a function such that when you differentiate that function you get 3 times the function.

(b) Can you think of a function such that when you differentiate that function you get the square of the function.

(c) Can you think of a function such that when you differentiate that function you get 2 times the square of the function.

(d) Can you think of a function such that when you differentiate that function you get 3 times the square of the function.

(e) None of the above.

(f) All of the above.

2. Which question is the following differential equation asking in words?

$$\frac{dy}{dx} = 3y^2$$

(a) Can you think of a function such that when you differentiate that function you get 2 times the square of the function you first thought of .

(b) Can you think of a function such that when you differentiate that function you get 3 times the function you first thought of .

(c) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of .

(d) Can you think of a function such that when you differentiate that function you get the square of the function you first thought of.

(e) None of the above.

3. Which question is the following differential equation asking in words?

$$\frac{dx}{dt} = 3x^2$$

(a) Can you think of a function such that when you differentiate that function you get $3t^2$

(b) Can you think of a function such that when you differentiate that function you get 3 times the function you first thought of .

(c) Can you think of a function such that when you differentiate that function you get the square of the function you first thought of.

(d) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of .

(e) Can you think of a function such that when you differentiate that function you get 2 times the square of the function you first thought of .

(f) None of the above.

4. Which question is the following differential equation asking in words?

$$\frac{dx}{dt} = 3x^2 + x^3$$

(a) Can you think of a function such that when you differentiate that function you get $3t^2 + t^3$

(b) Can you think of a function such that when you differentiate that function you get 3 times the function you first thought of plus that function cubed.

(c) Can you think of a function such that when you differentiate that function you get two times the square of the function you first thought of plus that function cubed.

(d) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of plus that function squared.

(e) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of .

(f) None of the above.

5. Match up the different differential equations with their corresponding questions, written in words.

(a) Can you think of a function such that when you differentiate that function you get $3t^2 + t^3$

(b) Can you think of a function such that when you differentiate that function you get 2 times the function you first thought of plus that function cubed.

(c) Can you think of a function such that when you differentiate that function you get two times the square of the function you first thought of plus that function cubed.

(d) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of plus that function squared.

(e) Can you think of a function such that when you differentiate that function you get 3 times the square of the function you first thought of .

(i)

$$\frac{df}{dt} = 2f^2 + f^3$$

(ii)

$$\frac{dx}{dt} = 4x^2$$

(iii)

$$\frac{dy}{dt} = 3t^2 + t^3$$

(iv)

$$\frac{dx}{dt} = 3x^2$$

(v)

$$\frac{dx}{dt} = 2x + x^3$$

2.3 Particular Solutions and General Solutions or Differentiating in Reverse

2.3.1 The difference between General Solutions and a Particular Solution

For this section we will concentrate on differential equations that look like

$$\frac{dy}{dx} = f(x),$$

that is equations with the derivative on the left and a function of only the independent variable on the right.

Examples of these type of differential equations are $\frac{dy}{dx} = 3x^2 + 2x$ or $\frac{dy}{dx} = 10x^4 - \sin(x)$, but with no solution functions, or ys here, on the right.

Example 1

The differential equation $\frac{dy}{dx} = 5$ is asking; can you think of a function $y(x)$ such that when you differentiate that function you get 5.

Just one answer is $y(x) = 5x$ as here $\frac{d}{dx}5x = 5$

If you know how to differentiate, answering these questions seems like we just need to apply the rules of differentiation in reverse.

This process is given the name anti-differentiation.

Anti-differentiation simply means apply the rules of differentiation in reverse, in order to solve our differential equations or answer our questions.

Seems simple so far, though there can be many answers to one question, just like the meaning of life.

Example 2

The differential equation $\frac{dy}{dx} = 2x$ is asking; what y do we differentiate to get $2x$?

We know $y(x) = x^2$ is an answer to this question as $\frac{d}{dx}x^2 = 2x$. So we have one solution to this differential equation.

But if we differentiate $y = x^2 + 1$ for instance, we also have $\frac{d}{dx}(x^2 + 1) = 2x$. This means that $y = x^2 + 1$ is also a solution to our differential equation. But so is $y = x^2 - 1$, $y = x^2 + 10$, $y = x^2 - 24.3$, $y = x^2 + 1000000$, $y = x^2 + \pi$ and $y = x^2 - \pi \times 1000000$.

There can be many solutions to just one differential equation.

If we differentiate any constant we get 0. So if we differentiate

$$y = x^2 + \text{any constant}$$

we get

$$\frac{d}{dx}(x^2 + \text{any constant}) = 2x.$$

So there are lots of answers to our question $\frac{dy}{dx} = 2x$. In fact there are an infinite number of them.

The most general solution to this differential equation is $y = x^2 + c$, where c is a constant.

The constant c, as used here, is called the constant of integration.

Since there may be lots of solutions to any one differential equation we use special names for each type.

The solution $y = x^2 + c$ is called the **general solution** of the differential equation $\frac{dy}{dx} = 2x$.

As it is the most general solution to the differential equation.
Any one of these solutions on there own, such as, $y = x^2$, $y = x^2 + 100$, $y = x^2 - \pi \times 10000000000000000$ are all still solutions to the differential equation.
Any one of these solutions is called a particular solution to the differential equation.
$y = x^2$ is a particular solution to the differential equation. $y = x^2 + 100$ is also a particular solution to the differential equation. $y = x^2 - \pi \times 10000000000000000$ is also a particular solution to the differential equation.

The solution $y = x^2$ is called a **particular solution** of the differential equation $\frac{dy}{dx} = 2x$.

As it is the just one solution to the differential equation.

2.3.2 Exercises

1. For the differential equation
$$\frac{dy}{dx} = 3x^2$$
which of the below are particular solutions?
   (a) $y = x^3$
   (b) $y = x^3 - 10000000$
   (c) $y = x^3 - \pi$
   (d) $y = x^2 + 20000000$
   (e) $y = x^3 - c$
   (f) None of the above.
   (g) All of the above.

2. For the differential equation
$$\frac{dy}{dx} = 4x^3$$
which of the below are particular solutions?
   (a) $y = x^4 + 10$
   (b) $y = x^4 + \pi$
   (c) $y = x^3 + c$
   (d) $y = x^3 - c$
   (e) $y = 4x^3 + c$
   (f) None of the above.
   (g) All of the above.

3. For the differential equation
$$X'(t) = \sin(t)$$
which of the below are general solutions?
   (a) $X(t) = cos(t) + c$
   (b) $X(t) = -cos(t) + 10$
   (c) $X(t) = -cos(t) + c$
   (d) $X(t) = cos(t) + 2d$
   (e) $X(t) = -cos(t) + 2d$
   (f) None of the above.
   (g) All of the above.

2.3.3 Particular Solutions and General Solutions in General

The most general solution to a differential equation is called **the general solution**.

For the differential equations with only the first derivative, in these notes, the general solution will involve a constant such as c above.

The constant c, as used here, is called the **constant of integration**.

Any single solution to a differential equation is called **a particular solution**.

The particular solution will **not** involve a constant of integration, such as c above.

Summary of Terminology For Differential Equations

$\dfrac{dy}{dx} = f(x)$ is called a differential equation. There are many other types.
$y = \displaystyle\int f(x)\,dx$ is called the general solution of the differential equation. The integral will involve a constant c say.
c is called the constant of integration.
If we evaluate c, for instance if $c = 0$ then $y = \displaystyle\int f(x)\,dx$, with $c = 0$, is called a particular solution of the differential equation.
The process of answering a question about the derivative of a function that we are using here is called anti-differentiation.

Example

$\dfrac{dy}{dx} = 5x^2$ is called a differential equation.
$y = \dfrac{5}{3}x^3 + c$ is called the general solution of the differential equation.
c is called the constant of integration.
If we evaluate c, for instance if $c = 0$ then $y = \dfrac{5}{3}x^3$ is called a particular solution of the differential equation. $y = \dfrac{5}{3}x^3 + 10$ is also a particular solution.

The process we are using of answering a question about the derivative of a function is called anti-differentiation.

2.3.4 Exercises

The following mathematical expressions, that are labelled (a) to (f), are **just part** of a solution to a problem involving a differential equation.
Match up each mathematical expression with the appropriate description in words, labelled (i) to (vi).

Please note: In general you should provide answers with descriptions of the solution in words as well as mathematics.

As an example of how to set out solutions in words and mathematics see **Ten Important Steps for Solving Modeling Questions Posed in Words**

1. For the differential equation $\dfrac{dy}{dx} = 2x$

   (a) $\dfrac{dy}{dx} = 2x$

   (b) $y = x^2 + c$

   (c) $y = x^2 + 5$

   (d) c

   (e) $c = 5$

   (f) $y = x^2$

   (i) Differential Equation

   (ii) Particular Solution

   (iii) General solution

   (iv) Particular Solution

   (v) Constant of Integration

   (vi) Evaluation of constant of integration

2. For the differential equation $\dfrac{dy}{dx} = 5x^4$

   (a) $c = 5$

   (b) $y = x^5 + c$

   (c) $y = x^5 + 5$

   (d) c

   (e) $\dfrac{dy}{dx} = 5x^4$

   (f) $y = x^5 + c$

   (i) General solution

   (ii) Particular Solution

   (iii) Differential Equation

   (iv) Constant of Integration

   (v) General Solution

   (vi) Evaluation of constant of integration

3. For the description of a differential equation, given by the following description;

   > "Can you think of a function such that when you differentiate that function with respect to t you get $\sin(t)$."

   match up each mathematical expression labelled (a) to (f) with the appropriate description in words, labelled (i) to (vi), if an appropriate description exists.

   (a) $X = \cos(t) + c$

   (b) $X = -\cos(t)$

   (c) $\dfrac{dt}{dx} = \sin(t)$

   (d) $X = -\cos(t) + c$

   (e) $\dfrac{dX}{dt} = \sin(t)$

   (f) $X = -\cos(t) + 15$

   (i) General solution

   (ii) Particular Solution

   (iii) Differential Equation

   (iv) Constant of Integration

   (v) Particular Solution

   (vi) Evaluation of constant of integration

4. For the description of a differential equation, given by the following description;

> "Can you think of a function such that when you differentiate that function with respect to x you get $x\,e^x + e^x$."

match up each mathematical expression labelled (a) to (f) with the appropriate description in words, labelled (i) to (vi), if an appropriate description exists.

(a) $\dfrac{dy}{dx} = x\,e^x + e^x + c$

(b) $Y = x\,e^x + c$

(c) $Y = x\,e^x - 42$

(d) $Y'(x) = x\,e^x + e^x + c$

(e) $Y'(x) = x\,e^x + e^x$

(f) $Y(x) = x\,e^x + e^\pi$

(i) General solution

(ii) Particular Solution

(iii) Differential Equation

(iv) Constant of Integration

(v) Particular Solution

(vi) Evaluation of constant of integration

Chapter 3

Modelling and Calculus 3 MAC 3 Solving some differential equations by using the concepts and interpretations of a differential equation to find a solution or an answer posed by the differential equation.

3.1 Solving Differential Equations of the Form $\dfrac{dy}{dx} = f(x)$ Using Differentiation in Reverse

The most important step in solving differential equations, at least at the beginning, is to understand what a differential equation represents and what it is asking.

The second most important step in starting to solve differential equations is understanding the terminology and notation used.

We now use these ideas to solve some simple differential equations.

Here we concentrate on differential equations of the form $\dfrac{dy}{dx} = f(x)$.

For these differential equations the right hand side is always some function of x, like $\dfrac{dy}{dx} = 2x$, $\dfrac{dy}{dx} = 3x^2$ etc:

To solve a differential equation of this form we are asked the question:

What function y, which is a function of x, has a derivative given by $f(x)$.

If you know various derivatives this may be just a matter of, remembering derivatives, or looking up **differentiation tables**.

Example 1

Find one solution of the differential equation

$$\frac{dy}{dx} = \cos(x).$$

From the **differentiation tables** $\dfrac{d}{dx}\sin(x) = \cos(x)$, so **a** solution is $y(x) = \sin(x)$.

Example 2

For the differential equation

$$F'(t) = 3e^{3t}$$

find one particular solution.

From the **differentiation tables** $\dfrac{d}{dt}e^{3t} = 3e^{3t}$, so **a** particular solution is $F(t) = e^{3t}$.

3.1.1 Exercises

By using the **differentiation tables** identify which one of the options are particular solution to the differential equations.

1.
$$\frac{dy}{dx} = 10$$

   (a) $y = 10x + c$
   (b) $y = 10x^2$
   (c) $y = 10x - c$
   (d) $y = 10x - \pi$
   (e) None of the above.

2.
$$\frac{dy}{dx} = 6x$$

   (a) $y = 6x^2$
   (b) $y = 6 \times x^2/2 + c$
   (c) $y = 3x^2 + c$
   (d) $y = 3x^2$
   (e) None of the above.

3.
$$\frac{dy}{dx} = 3 \times 5x^4$$

   (a) $y = 3x^5$
   (b) $y = 3 \times x^5/5$
   (c) $y = 3 \times x^5/5 + c$
   (d) None of the above.

4.
$$\frac{dy}{dx} = 6x^5$$

   (a) $y = 6x^6$
   (b) $y = 6x^6 + c$
   (c) $y = 6 \times x^6/5$
   (d) $y = x^6 + c$
   (e) None of the above.

5.
$$\frac{dy}{dx} = 10 \times 14x^{13}$$

   (a) $y = 10x^{14} + c\pi$
   (b) $y = 10x^{14} + c$
   (c) $y = 10x^{14} - 100000\pi$
   (d) $y = 10x^{14}$
   (e) all of the above
   (f) none of the above.

6.
$$\frac{dX}{dt} = e^t$$

(a) $y = e^x$

(b) $y = e^x + c$

(c) $y = e^t + c$

(d) $X = e^t + \pi$

(e) none of the above.

7.
$$Y'(x) = 5x^4 + 3x^2$$

(a) $Y(x) = x^5 + x^3$

(b) $Y(x) = x^5 + x^3 + 42$

(c) $Y(x) = x^5 + x^3 - e^{42\pi}$

(d) $Y(x) = x^5 + x^3 + c$

(e) none of the above.

8.
$$X'(t) = \frac{1}{t}$$

(a) $X'(t) = \ln(t)$

(b) $X(t) = \ln(t) + c$

(c) $X(t) = \ln(t) - e^{355\pi + \ln(42)}$

(d) $Y(x) = \ln(x) - 1$

(e) none of the above.

9.
$$\frac{dy}{dx} = \sec^2(x)$$

(a) $y = \sec(x)$

(b) $y = \sec^2(x)$

(c) $y = \tan(x) + c$

(d) $y(x) = \tan(x) - 53$

(e) none of the above.

10.
$$\frac{dX}{dt} = 2t\sec^2(t^2)$$

(a) $X(t) = \tan(t^2)$

(b) $X = \tan(t^2) - 51\pi$

(c) $X = \tan(t^2) + 10000000000000000000000000000$

(d) $X = \tan(t^2) + 0$

(e) none of the above.

3.2 Differentiation Tables

No.	General Rule	Example 1	Example 2	Example 3
(1)	$\frac{d}{dx}(\text{of any constant c}) = 0$	$\frac{d}{dx}1 = 0$	$\frac{d}{dx}10 = 0$	$\frac{d}{dx}\pi \times 1000000001 = 0$
(2)	$\frac{d}{dx}x = 1$	$\frac{d}{dt}t = 1$	$\frac{d}{dz}z = 1$	
(3)	$\frac{d}{dx}x^2 = 2x$	$\frac{d}{dt}t^2 = 2t$	$\frac{d}{df}f^2 = 2f$	
(4)	$\frac{d}{dx}x^3 = 3x^2$	$\frac{d}{dt}t^3 = 3t^2$	$\frac{d}{du}u^3 = 3u^2$	
(5)	$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}x^4 = 4x^3$	$\frac{d}{dx}x^{-1} = -1x^{-2} = -\frac{1}{x^2}$	$\frac{d}{dx}x^{16} = 16x^{15}$
(6)	$\frac{d}{dx}cf(x) = c \times \frac{df(x)}{dx}$	$\frac{d}{dx}10x = 10\times 1 = 10$	$\frac{d}{dx}2x^2 = 2 \times 2x = 4x$	$\frac{d}{dx}\pi x^{0.3} = \pi \times 0.3x^{-0.7}$
(7)	$\frac{d}{dx}e^x = e^x$	$\frac{d}{dt}e^t = e^t$	$\frac{d}{du}e^u = e^u$	$\frac{d}{dz}\pi \times e^z = \pi \times e^z$
(8)	$\frac{d}{dx}e^{ax} = ae^{ax}$	$\frac{d}{dt}e^{3t} = 3e^{3t}$	$\frac{d}{du}e^{10u} = 10e^{10u}$	$\frac{d}{dz}e^{-2.3z} = -2.3e^{-2.3z}$
(9)	$\frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x) = f'(x)e^{f(x)}$	$\frac{d}{dx}e^{x^2} = 2xe^{x^2}$	$\frac{d}{dx}e^{2x^3} = 6x^2e^{2x^3}$	$\frac{d}{du}e^{\pi u^{-6}} = -6\pi u^{-7}e^{\pi u^{-6}}$
(10)	$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\frac{d}{dt}3\ln(t) = 3 \times \frac{1}{t}$	$\frac{d}{dw}\pi\ln(w) = \frac{\pi}{w}$	
(11)	$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{f'(x)}{f(x)}$	$\frac{d}{dx}\ln(x^2) = \frac{1}{x^2} \times 2x$	$\frac{d}{dx}\ln(5x^4) = \frac{1}{5x^4} \times 20x^3$	$\frac{d}{dx}3\ln(\pi x^{-6}) = \frac{3}{\pi x^{-6}} \times -6\pi x^{-7}$

Differentiation Tables Continued

No.	General Rule	Examples
(12)	$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dt}\sin(t) = \cos(t)$
		$\frac{d}{dx}355\sin(x) = 355\cos(x)$
		$\frac{d}{dz} - \pi^2\sin(z) = -\pi^2\cos(z)$
(13)	$\frac{d}{dx}\sin(\,f(x)\,) = \cos(\,f(x)\,) \times \frac{d}{dx}f(x) = \cos(\,f(x)\,)\,f'(x)$	$\frac{d}{dx}\sin(2x) = \cos(2x) \times 2$
		$\frac{d}{dx}\sin(5x^2 - 4x) = \cos(5x^2 - 4x)\times(10x - 4)$
		$\frac{d}{dx}24\sin(e^{3x}) = 24\cos(e^{3x}) \times 3e^{3x}$
(14)	$\frac{d}{dx}\cos(x) = -\sin(x)$	$\frac{d}{dx}24\cos(x) = -24\sin(x)$
		$\frac{d}{dz}(-3\pi^5\cos(z)) = 3\pi^5\sin(z)$
		$\frac{d}{df}\pi e^5\cos(f) = -\pi e^5\sin(f)$
(15)	$\frac{d}{dx}\cos(\,f(x)\,) = -\sin(\,f(x)\,) \times \frac{d}{dx}f(x) = -\sin(\,f(x)\,)\,f'(x)$	$\frac{d}{dx}\cos(2x) = -\sin(2x) \times 2$
		$\frac{d}{dx}\cos(-4x^3 + 3x^2 - 5x) = -\sin(-4x^3 + 3x^2 - 5x) \times (-12x^2 + 6x - 5)$
		$\frac{d}{dx}24\cos(e^{-\pi x}) = -24\sin(e^{-\pi x}) \times -\pi e^{-\pi x} = 24\pi e^{-\pi x}\sin(e^{-\pi x})$

Differentiation Tables Continued

No.	General Rule	Examples
(16)	$\dfrac{d}{dx}\tan(x) = \sec^2(x)$	$\dfrac{d}{dx}\tan(t) = \sec^2(t)$
		$\dfrac{d}{dx}42\tan(x) = 42\sec^2(x)$
		$\dfrac{d}{dz}4\pi^2\tan(z) = 4\pi^2\sec^2(z)$
(17)	$\begin{aligned}\dfrac{d}{dx}\tan(\,f(x)\,) &= \sec^2(\,f(x)\,)\times\dfrac{d}{dx}f(x)\\ &= \sec^2(\,f(x)\,)f'(x)\end{aligned}$	$\dfrac{d}{dx}\tan(x^3) = \sec^2(x^3)\times 3x^2$
		$\dfrac{d}{dx}\tan(4x^3-3x+2) = \sec^2(4x^3-3x+2)\times(12x^2-3)$
		$\dfrac{d}{dx}-5692\pi^2\tan(e^{53\pi x}) = -5692\pi^2\sec^2(e^{53\pi x})\times 53\pi e^{53\pi x}$

3.3 Solving Differential Equations of the Form $\dfrac{dy}{dx} = f(x)$ Using Integration

A differential equation of the form $\dfrac{dy}{dx} = f(x)$ may be solved by using differentiation in reverse that is by anti-differentiation by asking:

What function $y(x)$ can be differentiated to give $f(x)$?

But we can also try to solve this type of equation by using integration.

We can "undo" the $\dfrac{d}{dx}$ on the left by integrating. If the function on the right, that is $f(x)$ can be integrated (over the interval of x) then we can find a solution to the differential equation.

We start with the differential equation

$$\frac{dy}{dx} = f(x).$$

We then integrate both sides of the equation with respect to the independent variable, here x.

Given certain conditions if we performing the same operation to both sides of the equation then the two sides of the equation will remain equal. This is analogous to multiplying both sides of an equation by 2, or taking the square of both sides of the equation. So long as we perform precisely the same operation to both sides of the equation the equality remains valid.

You must always integrate both sides of the equation with respect to the same variable.

$$\int \frac{dy}{dx}\, dx = \int f(x)\, dx.$$

Giving the solution;

$$y = \int f(x)\, dx.$$

For this special case then we can use integration or **integration tables** to find a solution for the differential equation.

Example 1

Question

Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2$$

Answer

Step 1 Integrate both sides of the differential equation with respect to the independent variable x,

$$\int \frac{dy}{dx}\, dx = \int x^2\, dx$$

Step 2 perform the integration,

$$\begin{aligned} y &= \int x^2\, dx \\ &= \frac{1}{3}\, x^3 + c, \end{aligned}$$

where c is a constant.

Step 3 State the answer **in words and mathematics**.

The general solution is

$$y = \frac{1}{3}\, x^3 + c.$$

Example 2

Question

Find the general solution to the differential equation

$$\frac{dX}{dt} = \frac{-2t}{3-t^2}$$

Answer

Step 1 Integrate both sides of the differential equation with respect to the independent variable t,

$$\int \frac{dX}{dt}\,dt = \int \frac{-2t}{3-t^2}\,dt$$

Step 2 perform the integration,

$$\begin{aligned} X(t) &= \int \frac{-2t}{3-t^2}\,dt \\ &= \ln|3-t^2| + c, \end{aligned}$$

where c is a constant.

> Note: here we are using the rule that says if we have a function on the bottom of a fraction and the derivative of that function on the top then the integral of that fraction is the natural log of the absolute value of that function or in the language of mathematics :
>
> $$\int \frac{f'(x)}{f(x)} dx = \ln(|f(x)|) + c$$
>
> So for our integral we need the $-2t$ on the top to use this rule.

Step 3 State the answer **in words and mathematics**.

The general solution is

$$X(t) = \ln|3-t^2| + c,$$

where c is a constant.

> **In General**:
> To solve a differential equation of the form $\frac{dy}{dx} = F(x)$ we **may** be able to find the integral of $F(x)$. Lets say $\int F(x)\,dx = f(x) + c$. The steps are:
>
> **Step 1** Integrate both sides of the differential equation
>
> $$\int \frac{dy}{dx}\,dx = \int F(x)\,dx$$
>
> **Step 2** perform the integration,
>
> $$\begin{aligned} y &= \int F(x)\,dx \\ &= f(x) + c, \end{aligned}$$
>
> where c is a constant.
>
> **Step 3** State the answer in words and mathematics.
>
> The general solution is
>
> $$y = f(x) + c.$$

3.3.1 Exercises

By integrating both sides of the differential equations below, find the most general solutions. The integration tables may be useful for finding the integrals.
Note for each of these exercises you should set out the solution indicating in words what you are doing at each stage. The explanation of what you are doing in words is often awarded marks in an examination.

1. $\dfrac{dy}{dx} = 3x^3$
2. $\dfrac{dy}{dx} = 2$
3. $\dfrac{dy}{dx} = 5x^6 - 16x^3$
4. $\dfrac{dy}{dx} = \pi x^2$
5. $\dfrac{dy}{dx} = \cos(x)$
6. $\dfrac{dy}{dx} = e^{5x}$
7. $\dfrac{dy}{dx} = 2x + \cos(x)$
8. $\dfrac{dy}{dx} = \cos(3x)$
9. $\dfrac{dy}{dx} = \cos(x^2) \times 2x$
10. $\dfrac{dy}{dx} = e^{x^2} \times 2x$
11. $\dfrac{dy}{dx} = \dfrac{1}{x^5} \times 5x^4 + \sin(3x^5) \times 15x^4$

3.4 Checking Your Solution

We can **always** check if a solution is correct by substituting the solution or answer into the differential equation (d.e.) and differentiating.

Example 1

To check the answer $y = \frac{1}{3}x^3 + c$ to the differential equation $\frac{dy}{dx} = x^2$ we simply substitute the answer into the differential equation.

Check

$$\begin{aligned} \text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) \\ &= \frac{1}{3} \times 3x^2 + \frac{d}{dx}c \\ &= x^2 + 0 \\ &= x^2 \\ &= \text{Right hand side of d.e.} \end{aligned}$$

So our solution works when we put it back into the differential equation.

> You should always check your answer by substituting the solution back into the differential equation. When answering questions in an exam it may be best to leave this step until the end, if time permits.

> If you are ever asked to **verify or check** if a solution satisfies a differential equation simply substitute the solution into the differential equation and show that it is satisfied. In many instances this will be **much simpler** than solving the differential equation.

3.4.1 Exercises

For each of the solutions below verify that the solution satisfies the differential equation
Note you are only required to substitute the solution into the equation and show that is works.

1. Solution $y(x) = \frac{3}{4}x^4$, differential equation $\frac{dy}{dx} = 3x^3$.
2. Solution $y(x) = 2x + c$, differential equation $\frac{dy}{dx} = 2$
3. Solution $y(x) = \frac{5}{7}x^7 - \frac{16}{4}x^4$, differential equation $\frac{dy}{dx} = 5x^6 - 16x^3$
4. Solution $y(x) = \pi\frac{x^3}{3} + c$, differential equation $\frac{dy}{dx} = \pi x^2$
5. Solution $y(x) = \sin(x) + 3$, differential equation $\frac{dy}{dx} = \cos(x)$
6. Solution $y(x) = \frac{1}{5}e^{5x}$, differential equation $\frac{dy}{dx} = e^{5x}$
7. Solution $y(x) = \frac{x^2}{2} + \sin(x)$, differential equation $\frac{dy}{dx} = x + \cos(x)$
8. Solution $y(x) = \frac{1}{3}\sin(3x)$, differential equation $\frac{dy}{dx} = \cos(3x)$
9. Solution $y(x) = \sin(x^2) - \pi$, differential equation $\frac{dy}{dx} = \cos(x^2) \times 2x$
10. Solution $y(x) = e^{x^2} + c$, differential equation $\frac{dy}{dx} = e^{x^2} \times 2x$
11. Solution $y(x) = x^5\cos(15x^4)$, differential equation $\frac{dy}{dx} = 5x^4\cos(15x^4) - 60x^8\sin(15x^4)$

3.5 Integration Tables

No.	General Rule	Example 1	Example 2
(1)	$\int k dx = kx + c$	$\int 2dx = 2x + c$	$\int 10\pi dx = 10\pi x + c$
(2)	$\int kx dx = k\frac{x^2}{2} + c$	$\int 3x dx = 3\frac{x^2}{2} + c$	$\int -5x dx = -5\frac{x^2}{2} + c$
(3)	$\int kx^2 dx = k\frac{x^3}{3} + c$	$\int -4x^2 dx = -4\frac{x^3}{3} + c$	$\int \pi x^2 dx = \pi\frac{x^3}{3} + c$
(4)	$\int kx^n dx = k\frac{x^{n+1}}{n+1} + c \quad n \neq -1$	$\int 2x^3 dx = 2\frac{x^4}{4} + c$	$\int \pi x^{-5} dx = \pi\frac{x^{-4}}{-4} + c = -\frac{\pi}{4x^4} + c$
		$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$	
(5)	$\int kf(x) dx = k\int f(x) dx$	$\int 5\, x^{16} dx = 5\int x^{16} dx = 5\,\frac{x^{17}}{17} + c$	
		$\int -\frac{\pi^2}{214} x^{5\pi} dx = -\frac{\pi^2}{214}\int x^{5\pi} dx = -\frac{\pi^2}{214}\frac{x^{5\pi+1}}{5\pi+1} + c$	
(6)	$\int \frac{1}{x}\, dx = \ln(\lvert x\rvert) + c$	interval of integration not across $x = 0$	
		$\int 5 \times \frac{1}{x}\, dx = 5\ln(\lvert x\rvert) + c$	$\int -\frac{10}{x}\, dx = -10\ln(\lvert x\rvert) + c$
(7)	$\int \frac{f'(x)}{f(x)} dx = \ln(\lvert f(x)\rvert) + c$	interval of integration not across $f(x) = 0$	
		$\int \frac{2x}{x^2}\, dx = \ln(\lvert x^2\rvert) + c$	$\int \frac{30x^5}{5x^6} dx = \ln(\lvert 5x^6\rvert) + c$
		$\int \frac{\cos(x)}{\sin(x)} dx = \ln(\lvert\sin(x)\rvert) + c$	$\int \frac{15e^{5x}}{3e^{5x}} dx = \ln(\lvert 3e^{5x}\rvert) + c$

Integration Tables Continued

No.	General Rule	Example 1	Example 2
(8)	$\int \cos(x)dx = \sin(x) + c$	$\int \cos(t)dt = \sin(t) + c$	$\int \pi^2 \cos(t)dt = \pi^2 \sin(t) + c$
(9)	$\int \cos(kx)dx = \frac{1}{k}\sin(x) + c$	$\int \cos(2x)dx = \frac{1}{2}\sin(2x) + c$	$\int \cos\left(\frac{1}{2}x\right) dx = 2\sin\left(\frac{1}{2}x\right) + c$
(10)	$\int \sin(x)dx = -\cos(x) + c$	$\int \sin(t)dt = -\cos(t) + c$	$\int 235\sin(t)dt = -235\cos(t) + c$
(11)	$\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + c$	$\int \sin(\pi x)dx = -\frac{1}{\pi}\cos(\pi x) + c$	$\int \sin\left(\frac{1}{2}x\right) dx = -2\cos\left(\frac{1}{2}x\right) + c$
(12)	$\int \cos(f(x)) \times f'(x)dx = \sin(f(x)) + c$	$\int \cos(x^2) \times 2xdx = \sin(x^2) + c$	
		$\int \cos(x^2+2x)\times(2x+2)dx = \sin(x^2+2x)+c$	
		$\int \cos(5e^{7x})\times 35e^{7x}dx = \sin(5e^{7x}) + c$	
(13)	$\int \sin(f(x)) \times f'(x)\,dx = -\cos(f(x)) + c$	$\int \sin(x^2) \times 2x\,dx = -\cos(x^2) + c$	$\int \cos(x^2) \times 2x\,dx = \sin(x^2) + c$
		$\int \sin(-x^3)\times(-3x^2)dx = -\cos(-x^3)+c$	$\int \sin(\ln(x))\times\frac{1}{x}dx = -\cos(\ln(x)) + c$

3.6 Solving Differential Equations of the Form $\dfrac{dy}{dx} = f(y)$

For a differential equation of the form

$$\frac{dy}{dx} = f(x),$$

if we can integrate $f(x)$ we can simple integrate both sides of the equation to find $y(x)$. For such equations see the section **Solving Differential Equation of the Form** $\frac{dy}{dx} = f(x)$ **Using Integration**.

Alternatively, if the right hand side of the differential equation involves y, as we are considering here, the differential equation is asking essentially quite a different question.

3.6.1 Differential Equation of the Form $\dfrac{dy}{dx} = y$

For instance, the differential equation:

$$\frac{dy}{dx} = y$$

is asking the question:

> Can you think of a function y, which is a function of x, such that if we differentiate that function we get back the the function we first thought of?

In other words, $\frac{dy}{dx} = y$ is asking what function is the same function when you differentiate it.

One answer is $y = e^x$ since

$$\frac{d}{dx}\,y = \frac{d}{dx}\,e^x = e^x = y = \text{the original function.}$$

The function $y(x) = e^x$ is a special function because it is, its own derivative.

But there are others.

What about $y = 2e^x$. Using the differentiation rules;

$$\frac{d}{dx}\,y = \frac{d}{dx}\,2e^x = 2\frac{d}{dx}\,e^x = 2e^x = y = \text{the original function.}$$

So $y = 2e^x$ is also its own derivative. In fact any function of the form $y = Ae^x$, where A is a constant, will have a derivative equal to the function we first started with as;

$$\frac{d}{dx}\,y = \frac{d}{dx}\,Ae^x = A\frac{d}{dx}\,e^x = Ae^x = y = \text{the original function.}$$

In fact a constant times the exponential function are the only functions with this property.

The functions $y(x) = Ae^x$, where A is a constant, are the only functions that are their own derivatives.

So $y(x) = Ae^x$ is the **general solution** to the differential equation $\frac{dy}{dx} = y$.

3.6.2 Exercises

1. Write down three particular solutions for the differential equation

$$\frac{dy}{dx} = y$$

2. Write down a sentence in words that describes what the following differential equation is asking in mathematics

$$\frac{dX}{dt} = X$$

3. Write down three particular solutions to the differential equation and hence the equivalent worded question in Question 2.

4. Use your answers to Question 3 to write down a general solution to the differential equation on Question 2.

5. Check your general solution in Question 4 by substituting your answer into the differential equation

$$\frac{dX}{dt} = X$$

6. Write down three particular solutions and the general solution to

$$Z'(y) = Z(y)$$

3.6.3 Differential Equation of the Form $\dfrac{dy}{dx} = ky$

The differential equation $\dfrac{dy}{dx} = 2y$ is asking the question:

What function y when you differentiate it gives twice the original function?

Such a function is $y(x) = e^{2x}$. Since

$$\frac{d}{dx}\,y = \frac{d}{dx}\,e^{2x} = 2e^{2x} = 2 \times y = \text{twice the original function.}$$

Another function that satisfies this is $y = 10e^{2x}$. Here

$$\frac{d}{dx}\,y = \frac{d}{dx}10e^{2x} = 10\frac{d}{dx}e^{2x} = 10 \times 2e^{2x} = 2 \times 10e^{2x} = 2 \times y = \text{twice the original function.}$$

That is, $y = 10e^{2x}$ a function, such that when we differentiate it, we get back two times the function we first started with.

In fact any function of the form $y(x) = Ae^{2x}$ will have the derivative equal to twice the function we first started with. And these are the only functions where $\dfrac{dy}{dx} = 2y$. Hence

> The functions $y(x) = Ae^{2x}$, where A is a constant, are the only function with their derivatives equal to twice the original functions.

Indeed if we differentiate $y = e^{kx}$, the derivative will be k times the original function;

$$\frac{d}{dx}\,y = \frac{d}{dx}e^{kx} = \frac{d}{dx}e^{kx} = k \times e^{kx} = k \times y = k \text{ times the original function.}$$

And finally any function of the form $y(x) = Ae^{kx}$ will have its derivative equal to k times the original function;

$$\frac{d}{dx}\,y = \frac{d}{dx}Ae^{kx} = A\frac{d}{dx}e^{kx} = Ak \times e^{kx} = k \times Ae^{kx} = k \times y = k \text{ times the original function.}$$

> The functions $y(x) = Ae^{kx}$, where A is a constant, are the only function, such that their derivatives are k times the original function.

> So $y(x) = Ae^{kx}$ is the general solution to the differential equation $\dfrac{dy}{dx} = ky$.

Example

Find the general solution of the differential equation $\dfrac{dy}{dx} = 15y$.

Here we want a function who's derivative is 15 times the original function. Such a function is $y(x) = e^{15x}$. So $y(x) = e^{15x}$ is **just one** particular solution to the differential equation. But this is not the only function that has its derivative equal to 15 times y. Any function of the form $y = Ae^{15x}$ will have its derivative equal to 15 times the original function. And since we are asked for the general solution we need to give the most general answer.

So $y(x) = Ae^{15x}$ is the general solution of the differential equation $\dfrac{dy}{dx} = 15y$.

Obviously we could just remember a formula for the general solution to this differential equation **but it is far more important to understand why something is a general solution, and what this means.**

> $\dfrac{dy}{dx} = 15y$ is a differential equation.
>
> $y(x) = e^{15x}$ is **just one** particular solution to the differential equation $\dfrac{dy}{dx} = 15y$.
>
> $y(x) = Ae^{15x}$, where A is a constant, is the general solution to the differential equation $\dfrac{dy}{dx} = 15y$.

3.6.4 Exercises

For the following questions the **differentiation tables** may be useful.

1. Which function when you differentiate it gives twice the original function?

   (a) $y = e^x$
   (b) $y = 2e^{2x}$
   (c) $y = e^{2t}$
   (d) $X = 2e^{2t}$
   (e) none of the above.

2. Which of the functions answers the question:

   Which function y when you differentiate it gives negative the original function?

   (a) $y = e^x$
   (b) $y = -e^x$
   (c) $y = e^{-t}$
   (d) $X = -e^{-t}$
   (e) none of the above.

3. Match up each of the questions with its differential equation.

   For this question you need to know/remember that the second derivative of y with respect to x is given by $\dfrac{d^2y}{dx^2}$. Likewise, by extension, the fourth derivative of y with respect to x is given by $\dfrac{d^4y}{dx^4}$.

   (a) Which function, when you differentiate it gives the same function?
   (b) Which function has a derivative which is twice the original function?
   (c) Which function is 5 times its derivative?
   (d) Which function is one fifth times its derivative?
   (e) Which function, when you differentiate it twice, gives minus the original function?
   (f) Which function, when you differentiate it 4 times, gives back the original function?

   (i) $\dfrac{d^2y}{dx^2} = -y$
   (ii) $\dfrac{d^4y}{dx^4} = y$
   (iii) $\dfrac{dy}{dx} = 5y$
   (iv) $\dfrac{dy}{dx} = y$
   (v) $\dfrac{dy}{dx} = 2y$
   (vi) $\dfrac{dy}{dx} = \dfrac{1}{5}y$

4. Match each question written in words labelled (a) to (f) to its differential equation labelled (i) to (vi).

(a) Which function, when you differentiate gives five times the original function?

(b) Which function, when you differentiate it give negative two times the original function?

(c) Which function when you differentiate it and add the original function gives you zero?

(d) Which function when you differentiate it and add three times the original function gives you zero?

(e) Which function when you differentiate it and subtract π times the original function gives you zero?

(f) Which function when you differentiate it and add it to π times the original function gives you zero?

(i) $\dfrac{dy}{dx} = -3y$

(ii) $\dfrac{dy}{dx} = -2y$

(iii) $\dfrac{dy}{dx} = -y$

(iv) $\dfrac{dy}{dx} = -\pi y$

(v) $\dfrac{dy}{dx} = 5y$

(vi) $\dfrac{dy}{dx} = \pi y$

3.6.5 How <u>Not</u> to Solve a Differential Equation of the Form $\dfrac{dy}{dx} = f(y)$

If the differential equation is of the form $\dfrac{dy}{dx} = f(x)$ and we can integrate $f(x)$ then we can find a solution by simply integrating both sides of the equation.

If, on the other hand, the differential equation has a function of y on the right hand side we can't just integrate.

> If the differential equation is of the form $\dfrac{dy}{dx} = f(y)$ we **can not** just integrate both sides of the equation to find the solution.

Example (of why we cant just integrate both sides of $\dfrac{dy}{dx} = f(y)$)

Lets say we have $\dfrac{dy}{dx} = y$.

We know that the general solution to this equation is $y(x) = Ae^x$.

We can *try* to integrate both sides of this equation with respect to x,

You must always integrate both sides of the equation with respect to the same variable.

which will give,

$$\int \frac{dy}{dx}\,dx = \int y\,dx.$$

A common mistake is then to **incorrectly** integrate the right hand side to give $y \times x + c$.

> But we **can not** integrate y with respect to x unless we know what y is, in terms of x.

For instance if we knew that $y(x) = Ae^x$ then we could happily substitute this into the equation to give;

$$\int \frac{dy}{dx}\,dx = \int y\,dx = \int Ae^x\,dx = Ae^x + c \quad \text{giving} \quad y = Ae^x + c.$$

For $c = 0$ this does give us the correct answer $y(x) = Ae^x$, which is it's own derivative. But sadly we have to know the original solution beforehand in order to solve the differential equation using this method.

We also note here that another, perhaps less common, mistake is to simplify the integral $\displaystyle\int y\,dx$ by incorrectly integrating with respect to y instead of x. That is we should not integrate $\displaystyle\int y\,dx$ to get $\dfrac{y^2}{2}$. Here we see that the dx is indeed very important in an integral.

The dx in the integral is called the differential and determines what the we are integrating with respect to.

> The dx in an integral such as $\displaystyle\int f(x)\,dx$ is called the differential.

> The differential dx determines what we are integrating with respect to; in this case x.

Here we see how important the differential is in an integral. For instance the integral $\displaystyle\int y(x)\,dx$ cannot be performed unless we know $y(x)$, that is y as a function of x. On the other hand $\displaystyle\int y\,dy$ can be found to be $\dfrac{y^2}{2}$ as here we are integrating with respect to y.

> The differential such as dx should **never ever**, **ever** be left off of an integral.

Initially these can be common mistakes in solving equations that involve $\dfrac{dy}{dx}$ and y. If these errors could be eliminated from many students exams these students would score more highly.

To solve the type of differential equation ($\dfrac{dy}{dx} = f(y)$) you can try to use the method of **separation of variables**.

Chapter 4

Modelling and Calculus 4 MAC 4 Relating written word descriptions of real world physical conditions to the description of the problem in the language of mathematics and using these descriptions to find solutions to the physical problem.

4.1 Finding Conditions in Questions

Often the environment and history of a problem will help to find a solution.

And so when a question is asked in words or mathematics we may have to interpret that question, and the words and mathematics, into mathematics that we can use to find a solution.

There are a number of ways that a question can give us information about the physical situation or ask us for an answer. Some of these are:

Type 1 Initial Condition The question may state certain conditions that have existed in the past, or at the beginning of the problem.

Type 2 Boundary Condition The question may tell us that one physical quantity has a value when another physical quantity has another value.

Type 3 Condition of Rate at a given time The question may tell us the value of a rate of change at a certain time.

Type 4 Condition of Rate The question may tell us the value of a rate of change when another physical quantity has another value.

Type 5 The question may ask us to find a physical quantity at a given time.

Type 6 The question may ask us to find a physical quantity when another quantity has a certain value.

Type 7 Common Sense Conditions We need to use some of our real world experience to place conditions on the variables. We don't want volumes or distances to take negative values. Or we may not want an answer of 3.14 people in a population.

Example 1

Question

A diving pool is the shape of a cube. This diving pool is drained for maintenance. The height of the water is given by H in metres. The pool drains at a rate proportional to $H + c$, where c is constant. Here then the differential equation is given by;

$$\frac{dH}{dt} = -kH + A,$$

where t is time measured in hours. The constant of proportionality is $-k$ and $A = -kc$. Why are the constants chosen in this way? Initially the water in the pool is 3 metres. After one hour the height of the water has droped to 2.5 metres. After two hours the height of the water is changing at a rate of 0.5 metres per hour.

1. Find the height of the water after three hours.
2. Find the time that it takes for all the water to drain out of the pool.

Answer

For this question the height of the water is changing with time. The height of the water H is the dependent variable; the height is the solution function. The time, in hours, is the independent variable. We want to find H as a function of t.

The sentence

> “ **Initially** the water in the pool is 3 metres.”

tells us what H was at the beginning. If we take $t = 0$ when the pool starts to drain then initially means at $t = 0$.

If we take time t equals zero at the beginning of the process then **initially** means $t = 0$.

This is called an initial condition and it is a condition of **Type 1** above.

The sentence

> “ **After** one hour the height of the water has droped to 2.5 metres.”

tells us the height after one hour; it tells us $H = 2.5$ at or when $t = 1$.

The words **after** or **at** or **when** may tell us the value of one quantity when another quantity has another value.

This is sometimes called a boundary condition and it is a condition of **Type 2** above.

The sentence

> “ **After** two hours the height of the water is changing at a rate of 0.5 metres per hour.”

tells us the **rate** of change of the height after two hours. This tells us $\frac{dH}{dt} = 0.5$ at $t = 2$.

The combination of words **rate** or **rate of change** together with **after** or **at** or **when** may tell us the derivative at a time t

The words **rate** or **rate of change** together with **after** or **at** or **when** may tell us $\frac{dy}{dt}$ at $t =$ something.

This is a condition of rate and it is a condition of **Type 3** above.

If the question said:

> " The height of the water was **changing at a rate** of 0.5 metres per hour **when** the water was 2 metres high."

This is telling us the rate of change of one quantity when the quantity has a given value. Here $\dfrac{dH}{dt} = 0.5$ when $H = 2$.

> The words **rate** or **rate of change** together with **when** may tell us $\dfrac{dy}{dt}$ at $y =$ something.

This is a condition of **type 4**.

The question asks us to:

> " Find the height of the water after three hours."

Which is asking us to find H when $t = 3$.

This is a question of **Type 5** above.

The question also asks us to:

> " Find the time that it takes for all the water to drain out of the pool."

Which is asking us to find the time t when the pool has emptied, that is when $H = 0$.

This is a question of **Type 6** above.

Example 2

Question

A partially full swimming pool is being filled by a water source, where the rate at which the water is entering the pool is decreasing with time. The height H in metres of water in the pool is given by;

$$\frac{dH}{dt} = k\sqrt{t},$$

where t is the time in hours after 11:00 am on a Thursday. If the pool **initially** satisfied $H(0) = 1.5$ and later the pool is filling at a rate given by $H'(4) = 6$.

Find

1. The constant of integration.
2. The constant of proportionality.
3. $H(4)$.
4. $H'(9)$.

Answer

For this question again the height of the water is changing with time. The height of the water H is the dependent variable; the height is the solution function. The time, in hours, is the independent variable. We want to find H as a function of t.

We are told the differential equation $\dfrac{dH}{dt} = k\sqrt{t}$. This equation is in the form $\dfrac{dy}{dx} = f(x)$ and we can simply integrate both sides with respect to t.

You must always integrate both sides of the equation with respect to the same variable.

$$\int \frac{dH}{dt} dt = \int k\sqrt{t} dt = k \int \sqrt{t} dt = k \int t^{1/2} dt.$$

So the general solution of the differential equation is

$$H(t) = k\,\frac{t^{3/2}}{3/2} + c = k\,\frac{2}{3}\,t^{3/2} + c$$

The statement

> " the pool **initially** satisfied $H(0) = 1.5$"

tells us what H was at the beginning. This time the question gives us the **initial** height as an equation. If we take $t = 0$ when the pool starts to fill then initially means at $t = 0$. Here the initial height is given as an equation. $H(0) = 1.5$ means that when the independent variable, here t, is zero then H is 1.5. Another way of writing this equation is $H(t = 0) = 1.5$. Yet another way is to simply say when $t = 0$ $H = 1.5$. You need to know all these forms of expressing a condition.

In general, for a function $y(x)$, the statement $y(a) = b$ means that when $x = a$ then $y = b$.

This is called an initial condition and it is a condition of **Type 1** above.

We can use this initial condition $H(0) = 1.5$ to find the constant of integration. Substituting into the differential equation we get $1.5 = k\,\frac{2}{3}\,t^{3/2} + c = c$. Hence the constant of integration $c = 1.5$.

The answer to part 1 of the question is then: The constant of integration is 1.5

The general solution is then $H(t) = k\,\frac{2}{3}\,t^{3/2} + 1.5$

The statement

> " later the pool is filling at a rate given by $H'(4) = 6$"

tells us H' at a given time. This time the question gives us the rate of change of height as an equation. The equation $H'(4) = 6$ tells us that at $t = 4$, $H'(t) = 6$ or at $t = 4$, $H' = 6$. Another way of writing this is to say that at $t = 4$ we have $\dfrac{dH}{dt} = 6$. This is telling us that after 4 hours the rate at of change of height of the water in the pool is 6 metres per hour. You need to know all these forms of expressing this condition.

In general, for a function $y(x)$, the statement $y'(a) = b$ means that when $x = a$ then $y'(x) = b$.

This is called a condition of rate and it is a condition of **Type 3** above.

Since this condition tells us something about the rate of change of H or H', to use this condition we need to firstly find an equation for H'. Differentiating the general solution for H gives

$$\frac{dH}{dt} = \frac{d}{dt}\left(k\,\frac{2}{3}\,t^{3/2} + 1.5\right) = k\,\frac{2}{3}\,\frac{3}{2}\,t^{3/2-1} = k\;t^{1/2}.$$

A much easier way of finding the derivative is to simply use the differential equation. $\dfrac{dH}{dt} = k\sqrt{t}$.

Substituting $t = 4$ into our equation for the derivative gives $H'(4) = k\sqrt{4} = 6$. Dividing both sides of this equation by 2 gives $k = 3$

The answer to part 2 of the question is then: The constant of proportionality is 3.

Substituting for k gives the general solution as $H(t) = 3\,\dfrac{2}{3}\,t^{3/2} + 1.5 = 2\,t^{3/2} + 1.5$.

Part 3 of the question is asking for $H(4)$ which is asking us to evaluate H at $t = 4$. To do this we simply substitute $t = 4$ into our general solution. Giving

$$H(t = 4) = H(4) = 2 \times 4^{3/2} + 1.5 = 2 \times 8 + 1.5 = 17.5.$$

The answer to part 3 is $H(4) = 17.5$

Part 4 of the question is asking for $H'(9)$ which is asking us to evaluate $H'(t)$ at $t = 9$, that is $\dfrac{dH}{dt}$ at $t = 9$. To do this we need to find the derivative of H. To do this we could differentiate $H(t)$ viz.;

$$\frac{dH}{dt} = \frac{d}{dt}\left(2\,t^{3/2} + 1.5\right) = 3\,t^{1/2} = 3\,\sqrt{t}.$$

Alternatively we can simply use the differential equation: The differential equation is

$$\frac{dH}{dt} = k\,\sqrt{t} \text{ substituting for } k \text{ gives} = 3\,\sqrt{t}$$

This gives us the derivative directly, without solving the differential equation by integration and then differentiating it again.

To find $H'(9)$ then we simply substitute $t = 9$ into our our formula for $\dfrac{dH}{dt}$. Giving

$$\frac{dH}{dt}_{\,t=9} = H'(9) = 3\,\sqrt{9} = 3 \times 3 = 9$$

The answer to part 4 is $H'(9) = 9$.

4.1.1 Exercises

For the following questions, a set of conditions are given in words and mathematics.

For each part **(a)**, **(b)**, **(c)**... write the condition as a set of mathematical equations (for example when $t = 0$ $H = 15$).

Note for these questions you do **not** need to find the constant of integration (c say) or the constant of proportionality (k say).

Note also that each part **(a)**, **(b)**, **(c)**... may give you a different value for the constant of integration and/or constant of proportionality.

1. During a period of no rain the height H of a river, in metres, drops according to the equation;

$$\frac{dH}{dt} = -0.15\,H,$$

where t is measured in hours after the rain stops.

(a) Initially the height of the river is 15 metres.

(b) One hour after the rain stops the height of the river is 13.5 metres.

(c) When the rain first stops falling the height of the river is 24 metres.

(d) Two days after the rain first stops falling the height of the river is 12.7 metres.

(e) At the time when the rain stops falling the height is 1550 mm.

(f) Three days after the rain first stops falling the height of the river is 2440 mm.

(g) $H(24) = 7$.

(h) $H(0) = 15.3$.

Note: Only one of these conditions will be needed to find the constant of integration in the general solution and thus find the particular solution.

2. The amount of medication in a patients body M, measured in milligrams, is given by the differential equation;

$$\frac{dM}{dt} = -kM,$$

where t is the time after the medication is administered, measured in minutes, and k is a constant.

(a) 150 milligrams of medication are administered.

(b) 150 minutes after the medication is given the amount in the blood is 45 mg

(c) Initially 250 milligrams of medication are given.

(d) Two days after the medication is given the amount in the blood is 17 mg.

(e) The rate of change of medication in the body is 12 mg/min when the amount of medication is 175 mg.

(f) Five hours after the medication is administered the rate of change of medication is 2 mg/min.

(g) One hour after the medication is given the rate of change of medication is given by 12.5 mg/min.

(h) $M(0) = 15.3$.

(i) $M'(60) = 3$.

(j) $M'(360) = 3.5$.

(k) $M(60) = 145$.

Note: You may need only two such conditions to find the constant of proportionality k, and the constant of integration in the general solution and thus find the particular solution.

3. The number of fish in a fishery holding pond, will increase at a rate proportional to the number of fish in the holding pond. The larger the population the quicker the fish will bread. This population is also being used as source of food such that fish are being removed at a constant cull rate R say. The number of fish in the population P measured in thousands of fish will be governed by the differential equation;

$$\frac{dP}{dt} = kP - R,$$

where t is the time after the holding pond is first established, measured in hours, and k is the reproduction rate constant.

(a) Initially the holding pond is stocked with 2.34 thousand fish.

(b) 3.5 hours after the holding pond is first established the fish population is 1324 fish.

(c) Initially the rate of change of fish in the population will be 2000 fish per hour.

(d) Two days after the holding pond is first established the fish population is decreasing at a rate of 0.524 thousand fish per hour.

(e) The rate of at which fish are being removed from the pond is (a constant) 500 fish per hour.

(f) 24 days 6 hours after establishing the holding pond there are no fish left.

(g) One day after establishment there are 22564 fish in the holding pond.

(h) $P(24) = 1000$.

(i) $P'(0) = 2$.

(j) $P'(24) = 2.0$.

(k) $P(150) = 0$.

Note: You may need only three such conditions to find; the production rate constant k, the cull rate constant R, and the constant of integration to find the particular solution from the general solution.

4.2 Ten Important Steps for Solving Modelling Questions Posed in Words

Here we work through an example showing how to identify the important information in the question or problem. We identify the key words or phrases in the question that provide us with information about the model and key words that provide us with information about physical environment, information about what has happened in the past, and what is being asked in the question.

We also identify important steps in the solution of the problem and provide guidelines for how the solution should be set out; including the wording that should be used in your solution.

There are other ways to solve every problem and the solution provided is just an example of how the answer may be set out.

Many answers to modeling and calculus problems will use some or all of the steps shown. Many answers to these modeling problems will use the steps shown and many more. They are a good place to start when learning modelling and differential equations.

Example

A hot pie is taken straight from the microwave, and put in a cold room that is at zero degrees Celsius.

The pie **cools** at a **rate proportional to** the **temperature of the pie**.

If the pie is **initially** at 86°C and takes 152 minutes to cool to half the temperature (in °C)

(i) Find the **general solution** to this problem.

(ii) Find the **particular solution** to this problem.

(iii) What temperature will the pie be **after** 10 hours?

(iv) **How long** does it take for the pie to be edible **at** 23°C or below?

This example has 10 important steps for solving a modelling question.

Step 1 Identify the variables in the question and give them names and units.

(a) From the question identify what will change.

Here the temperature is changing as time changes.

(b) Give this a name or symbol.

Let the temperature of the pie be T.

(c) Decide what this is measured in.

T is measures in °C (from the question).

(d) What does T change with?

Temperature changes over time.

(e) Decide what this is measured in?

Time is measured in minutes (from the question).

Here T is the dependent variable. We want to find the temperature as a function of time. So T is the solution function and t is the independent variable.

We want to find T (Temperature in °C) as a function of t (in minutes).
We want $T = T(t)$.

Step 2 From the question use the wording to put together a differential equation for T and t.

From the question **Rate** here means $\dfrac{d}{dt}$

The **rate of change of temperature** will be $\dfrac{dT}{dt}$

If the rate is **proportional** to the temperature then

$$\frac{dT}{dt} \propto T \quad \text{or} \quad \frac{dT}{dt} = KT,$$

where K is some constant.

Now, as indicated in the question, the pie **cools** at a **rate proportional to** the **temperature of the pie**. This means that the temperature of the pie will decrease quicker the hotter the pie is.
If the pie is cooling then the rate of change of T will be negative.
Really hot pies' temperatures will decrease quickly. Pies that are just a little warm will still cool down but their temperature will decrease more slowly. Really hot pies will have a large negative $\dfrac{dT}{dt}$ Warm pies will still have a negative $\dfrac{dT}{dt}$ but this will be less negative, indicating the temperature is still decreasing but less rapidly.

Since the pie is cooling the temperature will be **decreasing** so $\dfrac{dT}{dt}$ will be **negative**. So we have:

$$\frac{dT}{dt} = -\,kT,$$

where k here will be a positive constant ($k = -K$).

k is the **constant of proportionality**.

k should not be confused with the constant of integration that is will be introduced when we integrate the differential equation.

The differential equation is then

$$\frac{dT}{dt} = -\,kT$$

We note here that if the room was not at a constant temperature of 0°C then the pie would **cool** at a **rate proportional to** the difference between the temperature of the pie and the room.

If T_{Room} is the temperature of the room (held constant) then the differential equation for Newton law of cooling is

$$\frac{dT}{dt} = -\,k(T - T_{\text{Room}})$$

Step 3 Pose the differential equation as a mathematical question in words.
This step can help understand what the problem is asking and what the differential equation represents

The differential equation is

$$\frac{dT}{dt} = -kT$$

The differential equation is asking

> Can you think of a function T which is a function of t such that when you differentiate that function you get k times the original function?

Step 4 Find the general solution of the differential equation.

Use your knowledge of; differentiation backwards (anti-differentiation), or integration, or separation of variables, or another technique to solve the differential equation.

The function $y(x) = e^{-kx}$ has derivative $\frac{d}{dx}\, y = \frac{d}{dx} e^{-kx} = -ke^{-kx} = -k \times e^{-kx} = -k \times y(x)$.
So we know that the derivative of this function is $-k$ times the original function, and hence the function will solve the differential equation.

So $T(t) = e^{-kt}$ is **a** solution to the differential equation.

In general any function $T(t) = Ae^{-kt}$, where A is a constant, will solve the differential equation $\frac{dT}{dt} = -kT$.

Hence $T(t) = Ae^{-kt}$ is the general solution to the differential equation.

Hence $T(t) = Ae^{-kt}$ is the general solution to the differential equation.

Dont get the constants mixed up here

k is the constant of proportionality.
It comes from constructing the differential equation from the question.

A is the constant of integration.
It comes from integrating, or using anti-differentiation to find the general solution of the differential equation.

This answers part **(i)** of the question.

Step 5 From the question, pick out any information in words that will allow you to find the constants in our general solution.

The general solution

$$T(t) = Ae^{-kt}$$

has two constants (k and A), and we will need two sets of conditions from the question to find them.

The question says

"The pie is **initially** at 86°C".

Here **initial** means when we start to measure time.

So this tells us that at $t = 0$, $T = 86$.

The question also says

"...and takes 152 minutes to cool to half the temperature".

So after 152 minutes the temperature will be 43°C or $T = 43$.
So at $t = 152$, $T = 43$.

At $t = 0$ $T = 86$ is called an **initial condition**.

At $t = 152$, $T = 43$ is sometimes called a **boundary condition**.

Step 6 Use the boundary and initial conditions from the question to find a particular solution.

Substitute $t = 0$ and $T = 86$ into $T(t) = Ae^{-kt}$ gives:

$$86 = Ce^{-k\times 0} = Ae^0 = A \times 1 = A$$

Hence $A = 86$.

We then substitute this value of A back into our general solution to find the particular solution.

You must substitute the value of the constant of integration back into the general solution to find the particular solution.

State the particular solution

The particular solution is then $T(t) = 86e^{-kt}$.

This answers part **(ii)** of the question.

Step 7 Use the condition in the question to find the constant of proportionality.

We now use the second condition to find k.
Substituting $t = 152$ and $T = 43$ into the particular solution. We get

$$T(152) = 43 = 86e^{-k\times 152}.$$

To find k we:

$\div$ both sides by 86	$\dfrac{43}{86} = e^{-k\times 152}$
To get k out of the power take ln of both sides	$\ln\left(\dfrac{43}{86}\right) = \ln\left(e^{-k\times 152}\right) = -k \times 152$
$\div$ both sides by -152	$\dfrac{1}{-152}\ln\left(\dfrac{43}{86}\right) = \dfrac{-k\times 152}{-152} = k$
Work out the value of k on the calculator	$k \approx 0.00456$

Substitute this $k = 0.00456$ into the particular solution $T = e^{-kt}$, and state the answer.

> The particular solution for the conditions described in the problem is
>
> $$T(t) = 86e^{-0.00456t}$$

This answers part **(ii)** of the question with the constant of proportionality evaluated.

Step 8 From the wording of the question work out what the question is asking in mathematics.

Part **(iii)** of the question asks:

What will be the temperature of the pie after 10 hours?

Since we are using t measured in minutes we must change 10 hours to minutes.
To do this we use the **unit conversion factor**. Here we want to convert hours to minutes, and we know 1 hour is equivalent to 60 minutes. So here our **unit conversion factor** is $\dfrac{60\,\text{minute}}{1\,\text{hour}}$.

To change the units of a quantity from unit_1 to unit_2, where we know that A unit_1 is equivalent to B unit_2 we multiply by the **unit conversion factor** given by;

$$\textbf{unit conversion factor} = \frac{B\ \text{Unit}_2}{A\ \text{Unit}_1}$$

Examples

The **unit conversion factor** for converting minutes into hours $\dfrac{1\,\text{hour}}{60\,\text{minute}}$

The **unit conversion factor** for converting metres into kilometres is $\dfrac{1\,\text{kilometre}}{1000\,\text{metre}}$

The **unit conversion factor** for converting degrees into radians is $\dfrac{2\pi\,\text{radian}}{360\,\text{degree}}$

The **unit conversion factor** for converting radians into degrees is $\dfrac{360\,\text{degree}}{2\pi\,\text{radian}}$.

The length of time 60 minutes is equivalent to 1 hour so our **unit conversion factor** $\dfrac{60\,\text{minutes}}{1\,\text{hour}}$ is equal to 1. As an equation;

$$\textbf{unit conversion factor} = \frac{60\,\text{minutes}}{1\,\text{hour}} = 1$$

Hence if we multiply any quantity by the appropriate **unit conversion factor** this is equivalent to multiplying that quantity by unity. As we know multiplying a quantity by unity does not change the amount of that quantity.

Multiplying a quantity by the appropriate **unit conversion factor** does not change the amount of the quantity under consideration only the units in which the quantity is expressed.

In this instance to express our 10 hours in minutes we have;

$$10\,\text{hour} = 10\,\text{hour} \times \frac{60\,\text{minute}}{1\,\text{hour}} = 600\text{minute}.$$

Be careful to change all units to the units of the variables here T and t.

Step 9 Calculate the answer in mathematics.

Substituting $t = 600$ into $T(t) = 86e^{-0.00456\times t}$ gives

$$\begin{aligned} T &= 86e^{-0.00456\times 600} \\ &= 86e^{-2.736} \\ &= 86 \times 0.064829 \\ &= 5.5753 \end{aligned}$$

Step 10 State the answer in words including units.

If you just give $T = 5.5753$ this provides little information who doesn't know all of your working.

Is this the temperature or does T mean time? It could mean after 5.57 hours to an outsider.

State the answer in words including units.

The pie will cool to 5.5753°C after 10 hours.

This answers part **(iii)** of the question.

We now answer part **(iv)** of the question. To do this we work through **Step 8**, **Step 9** and **Step 10** again as we did for part for part **(iii)**.

Step 8 for part **(iv)** From the wording of the question work out what the question is asking in mathematics.

Part **(iv)** of the question asks:

How long does it take for the pie to be edible at 23°C or below?

Since we are measuring time from $t = 0$ when the pie is taken from the oven. We want to find the time t (in minutes) when the temperature T had dropped to 23°C (which is already in °C).

We want t when $T = 23$.

Always check the units when information is given in the question and when you are stating an answer.

Step 9 Calculate the answer in mathematics.

Substituting $T = 23$ into $T(t) = 86e^{-0.00456\times t}$ gives

$$23 = 86e^{-0.00456\times t}$$

We need to get t on its own. To do this we:

÷ both sides by 86	$\frac{23}{86} = e^{-0.00456\times t}$
To get t out of the power take ln of both sides	$\ln\left(\frac{23}{86}\right) = \ln\left(e^{-0.00456\times t}\right) = -0.00456 \times t$
÷ both sides by -0.00456	$\frac{1}{-0.00456}\ln\left(\frac{23}{86}\right) = \frac{-1.318853}{-0.00456} = t$
Work out the value of t on the calculator	$t \approx 289.22$

Step 10 State the answer in words including units.

Giving the answer in words allows someone who is unfamiliar with your choice of symbols or units to understand your answer.
T, t, x or y means little to someone who doesn't know the mathematics. Also an outsider will not know which units you have chosen.

State the answer in words including units.

The time for the pie to cool to 23°C is 289.22 minutes.

This answers part **(iv)** of the question.

Chapter 5

Answers to Selected Exercises

5.1 Answers to Exercises 1.1.2

Question 1

The rate of increase of temperature of the sausage may be represented by $\frac{dT}{dt}$, where T is the temperature of the sausage.

Question 2

The rate of change of concentration of salt may be represented by $\frac{dS}{dt}$, where S is the concentration of salt.

Question 3

The rate at which a human body produces insulin may be represented by $\frac{dI}{dt}$, where I is the insulin in the body.

Question 4

The rate at which fuel is taken from the fuel tank and fed into the engine may be represented by $\frac{dF}{dt}$ where F is the total volume of fuel being taken from the fuel tank and fed to the engine.

If the same amount of fuel is being fed to the engine per unit time the car will stay traveling at the same speed; that is if $\frac{dF}{dt}$ is constant then the speed of the car will remain the same.

Here there is another rate in the question. How fast the car is traveling may be represented by the rate at which the car changes its distance along a road; which may be represented by $\frac{dD}{dt}$, where D represents the distance of the car along the road.

Question 5

The rate at which $\frac{dF}{dt}$ changes will then be represented by $\frac{d}{dt}\frac{dF}{dt}$.

If the rate at which fuel is fed into the engine is simply represented by R then the rate of change of the rate at which fuel is fed to the engine will be represented by $\frac{dR}{dt}$.

The acceleration of the car is also represented by a derivative, it is the rate of change of velocity. So acceleration is given by $\frac{dv}{dt}$, where here the velocity v is the velocity along a straight road. More generally velocity in any direction is written as a vector $\mathbf{v}$ though here we will stick with scalar quantities.

Even the velocity is a derivative. The velocity v here is the rate of change of displacement along a (here) straight road. And is written as $\frac{dD}{dt}$ in the language of mathematics, where D is the distance along the straight road.

Question 6

The rate at which the depth of water drops in the pool may be represented by $\frac{dD}{dt}$, where D represents the depth of water in the swimming pool. If the rate at which the depth of water drops is dependent on the depth of water then $\frac{dD}{dt}$ will depend on D. Another way of saying this is that $\frac{dD}{dt}$ will be some function of D. Or $\frac{dD}{dt} = f(D)$; but more about this in the next section.

Question 7

The rate of change of the volume of the balloon may be represented in the language of mathematics as $\frac{dV}{dt}$ this mathematical expression represents the rate of change of a quantity and the quantity itself will be V the volume of the balloon. If the rate of decrease of volume of the balloon is dependent on the diameter D say then we can write in mathematics $\frac{dV}{dt} = f(D)$. But again we will explain this more thoroughly in the next section.

5.2 Answers to Exercises 1.2.3

Question 1

the rate of change of volume of the balloon is the derivative $\frac{dV}{dt}$.

the of volume of the balloon is the quantity V.

is equal to represents the equality or =.

2000 **litres per minute** tells us the rate of change.

The differential equation may be given by $\frac{dV}{dt} = 2000$ **you do not have to give this to answer the question**.

Question 2

The rate of change of temperature of a cold beer is the derivative $\frac{dT}{dt}$.

the temperature of a cold beer is the quantity T.

is equal to represents the equality or =.

the difference in temperature of the room and the beer tells us the rate of change.

The differential equation may be given by $\frac{dT}{dt} = R - B$ **you do not have to give this to answer the question**.

Question 3

The rate at which a swimming pool is filled is the derivative $\frac{dV}{dt}$.

the volume of water in the pool is the quantity V.

is equal to represents the equality or =.

a constant times the opening in the tap tells us the rate of change.

The differential equation may be given by $\frac{dV}{dt} = \text{constant} \times \text{Opening}_{\text{tap}}$ **you do not have to give this to answer the question**.

Question 4

the rate of change of the speed is the derivative $\frac{dv}{dt}$.

the speed of the bowling ball (downwards) is the quantity v.

is increasing by represents the equality or =.

9.8 metres per second every second tells us the rate of change.

The differential equation may be given by $\dfrac{dv}{dt} = 9.8$ **you do not have to give this to answer the question**.

Question 5

the rate of change of volume of the balloon is the derivative $\dfrac{dV}{dt}$.

the volume of the balloon is the quantity V.

is equal to represents the equality or $=$.

a constant times the volume of the balloon tells us the rate of change.

The differential equation may be given by $\dfrac{dV}{dt} = \text{constant} \times V$, here the constant will be negative **you do not have to give this to answer the question**.

Question 6

The rate of change of concentration of alcohol in the blood stream of a person is the derivative $\dfrac{dC}{dt}$.

the concentration of alcohol in the blood stream of a person is the quantity C.

is constant tells us about the equality $=$ (as well as other things).

is constant also tells us the rate of change.

The differential equation may be given by $\dfrac{dC}{dt} = \text{constant}$, here the constant will be negative as C decreases **you do not have to give this to answer the question**.

Question 7

The change of speed of an object falling to Earth will increase at a constant rate is another way of describing the derivative $\dfrac{dS}{dt}$.

the speed of an object falling to Earth (taken as downwards) is the quantity S.

will increase at a constant rate tells us about the equality $=$ (as well as other things).

will increase at a constant rate also tells us the rate of change.

The differential equation may be given by $\dfrac{dS}{dt} = 9.8$, **you do not have to give this to answer the question**.

Question 8

The rate of change of the wind-speed describes the derivative $\dfrac{dw}{dt}$.

the the wind-speed (where the wind-speed is positive if it is onshore and negative if is or offshore) is the quantity w.

will be dependent on tells us about the equality $=$ (as well as other things).

the difference in the temperature of the land and the ocean tells us the rate of change.

The differential equation may be given by $\dfrac{dw}{dt} = f(T_{\text{land}} - T_{\text{ocean}})$, here the constant will be negative as C decreases **you do not have to give this to answer the question**.

Question 9

the rate at which a small shark population increases describes the derivative as well as other thing (increases) $\dfrac{dP}{dt}$.

shark population is the quantity P.

will equal tells us about the equality $=$.

a constant times the number of fish in their habitat tells us the rate of change.

The differential equation may be given by $\dfrac{dP}{dt} = \text{constant} \times F$, here the constant will be positive as P will increase the larger F **you do not have to give this to answer the question**.

5.3 Answers to Exercises 1.3.2

Question 1

Yes, diameter $= 2 \times$ radius

Question 2

Yes, circumference $= \pi \times$ diameter

Question 3

Yes, circumference $= \pi \times$ diameter

Question 4

Yes, perimeter $= 4 \times$ side-length

Question 5

No, area $=$ side-length$^2 = 1 \times$ side-length^2. There is a constant 1 in the equation but the squared (^{2}) in the equation means the two quantities are not proportional.

Question 6

No, area $= \pi \times$ radius^2. Even though there is the constant π in the equation the squared (^{2}) in the equation means the two quantities are not proportional.

Question 7

Generally the height of a person and the weight of a person are not proportional. Even though a taller person may be heavier and a short person on average will be lighter there is no hard and fast single equation which relates the heigh and weight of people. If these two quantities were always proportional we could predict the weight of a person from their height, which in general we cannot.

Question 8

Again, generally speaking the size of a class will not determine how many people pass that class. If there is a very good year in a class with many hardworking students then a larger proportion should pass that class.

Question 9

Yes, area $= \pi \times$ radius^2. Note here the area and radius^2 are related only the constant π.

Question 10

The side length of a cube s that (just) contains a sphere of radius r will be $s = 2 \times \sqrt{3}r$. So the volume of the sphere V_{sphere} will be $V_{\text{sphere}} = \frac{4}{3}\pi r^3$ and the volume of the cube V_{cube} will be $V_{\text{cube}} = (2\sqrt{3}r)^3 = 8 \times 3\sqrt{3}r^3$.

Hence

$$V_{\text{cube}} = 24\sqrt{3}r^3 = 24\sqrt{3}\frac{3}{4\pi}\frac{4}{3}\pi r^3 = 24\sqrt{3}\frac{3}{4\pi}V_{\text{sphere}} = \frac{18\sqrt{3}}{\pi}V_{\text{sphere}}.$$

So the volumes of the cube and the sphere are related by $V_{\text{cube}} = \frac{18\sqrt{3}}{\pi}V_{\text{sphere}}$ so they are indeed proportional.

5.4 Answers to Exercises 1.3.4

Question 1

radius $= \frac{1}{2}$diameter, so the constant of proportionality is $\frac{1}{2}$.

Question 2

diameter $= \frac{1}{\pi}$circumference, so the constant of proportionality is $\frac{1}{\pi}$.

Question 3

side-length $= \frac{1}{4}$perimeter, so the constant of proportionality is $\frac{1}{4}$.

Question 4

radius $= \frac{1}{2\pi}$circumference, so the constant of proportionality is $\frac{1}{2\pi}$.

Question 5

volume$_{\text{cube}} = 1 \times$ side-length^3, so the constant of proportionality is 1.

Question 6

volume$_{\text{sphere}} = \frac{4}{3}\pi$radius^3, so the constant of proportionality is $\frac{4}{3}\pi$.

Question 7

An inch is defined to be inch = 24.5millimetre exactly, so the constant of proportionality is 24.5.

Question 8

A litre is defined to be 100 mm cubed. That is it is the volume contained in a cube that is 10 cms by 10 cms by 10 cms.
This means there are $10 \times 10 \times 10$ litres in a cubic metre.

In other words 1000 litres = 1cubic metre or 1 litres $= \frac{1}{1000}$cubic metre, so the constant of proportionality is 0.001.

Question 9

side-length $= \frac{1}{\sqrt{2}}$diagonal, so the constant of proportionality is $\frac{1}{\sqrt{2}}$.

Question 10

area$_{\text{circle}} = \pi$radius^2 and area$_{\text{square}} = (2\text{radius})^2$. Hence area$_{\text{square}} = \frac{4}{\pi}$area$_{\text{circle}}$, so the constant of proportionality is $\frac{4}{\pi}$.

Question 11

The side length of a cube s that (just) contains a sphere of radius r will be $s = 2 \times \sqrt{3}r$. So the volume of the sphere V_{sphere} will be $V_{\text{sphere}} = \frac{4}{3}\pi r^3$ and the volume of the cube V_{cube} will be $V_{\text{cube}} = (2\sqrt{3}r)^3 = 8 \times 3\sqrt{3}r^3$.

Hence

$$V_{\text{cube}} = 24\sqrt{3}r^3 = 24\sqrt{3}\frac{3}{4\pi}\frac{4}{3}\pi r^3 = 24\sqrt{3}\frac{3}{4\pi}V_{\text{sphere}} = \frac{18\sqrt{3}}{\pi}V_{\text{sphere}}.$$

So the volumes of the cube and the sphere are related by $V_{\text{cube}} = \frac{18\sqrt{3}}{\pi}V_{\text{sphere}}$ and the constant of proportionality is $\frac{18\sqrt{3}}{\pi}$.

Question 12

Many students would choose 100 % of the students to pass the class, if this were the case

$$\text{no. of students passing the class} = \text{no. of students in the class.}$$

So here the constant of proportionality would be 1.

Even if many students would choose all to pass, in order for "passing" the maths class to be worthy of some recognition, many students may agree that just enrolling should not be sufficient to pass.

Question 13

If we choose only 10 % of students to fail a maths class then

$$\text{no. of students failing the class} = \frac{10}{100}\text{no. of students in the class} = \frac{1}{10}\text{no. of students in the class,}$$

in this case the constant of proportionality would be 0.1.

If everyone passed the class then

$$\text{no. of students failing the class} = 0 = 0 \times \text{no. of students failing the class.}$$

We may be tempted to conclude that the constant of proportionality is 0. However the constant of proportionality is common defined to exclude 0. Under such a definition, in this case of everyone passing, the two quantities would be determined to be **not** proportional.

Question 14

area $= \pi \times \text{radius}^2$, so the constant of proportionality is π.

5.5 Answers to Exercises 2.1.4

Question 1

Solution function is y. The independent variable is x.

Question 2

Solution function is y. The independent variable is x.

Question 3

Solution function is y. The independent variable is x.

Question 4

Solution function is y. The independent variable is x.

Question 5

Solution function is x. The independent variable is y.

Question 6

Solution function is x. The independent variable is t.

Question 7

Solution function is x. The independent variable is t.

Question 8

Solution function is z. The independent variable is y.

Question 9

Solution function is z. The independent variable is x.

Question 10

There are two possible answers to this question.

As the equation stands or multiplying throughout by $\frac{dy}{dx}$ to give $1+3y\frac{dy}{dx}=0$ gives the solution function as y and the independent variable is x.

Alternatively and just as correctly we could use the property that for well behaved functions $y(x)$, $\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$.

We can then rearrange the differential equation to be $\frac{dx}{dy}+3y=0$ in which case the solution function is x and the independent variable would be y.

Question 11

Solution functions are x and y. The independent variable is t.

Question 12

Solution function is X. The independent variable is t.

Question 13

Solution function is X. The independent variable is t.

Question 14

Solution function is X. The independent variable is t.

Question 15

Solution function is X. The independent variable is Z.

Question 16

Solution function is Y. The independent variable is Z.

Question 17

Solution function is X. The independent variable is t.

Question 18

Solution functions are X and Y. The independent variable is t.

Question 19

Solution function is X. The independent variable is t.

Question 20

Solution function is X. The independent variable is t.

5.6 Answers to Exercises 2.2.2

Question 1

Can you think of a function y, which is a function of x, such that when you differentiate that function you get 2.

Question 2

Can you think of a function y, which is a function of x, such that when you differentiate that function you get $2x$.

Question 3

Can you think of a function y, which is a function of x, such that when you differentiate that function you get $\sin(x)$.

Question 4

Can you think of a function y, which is a function of x, such that when you differentiate that function you get $3x^2$.

Question 5

Can you think of a function y, which is a function of x, such that when you differentiate that function you get e^x.

Question 6

Can you think of a function y, which is a function of x, such that when you differentiate that function you get $x\sin(x)$.

Question 7

Can you think of a function y, which is a function of x, such that when you differentiate that function and add $2x$ you get 3.

Question 8

Can you think of a function X, which is a function of t, such that when you differentiate that function you get $3t^2$.

Question 9

Can you think of a function Y, which is a function of z, such that when you differentiate that function you get $\sin(z) + z$.

Question 10

Can you think of a function W, which is a function of t, such that when you differentiate that function and add $\sin(t)$ to it you get 15.

Question 11

Can you think of a function y, which is a function of x, such that when you differentiate that function and then multiply the derivative by x you get 3.

Question 12

Can you think of a function y, which is a function of x, such that when you differentiate that function and then multiply the derivative by $\sin(x)$ you get $x\sin(x)$.

Question 13

Can you think of a function y, which is a function of x, such that when you differentiate that function and then multiply the derivative by e^x you get e^{2x}.

Question 14

Can you think of a function y, which is a function of x, such that when you differentiate that function and then multiply the derivative by itself (that is square it) you get $(3x^2 + 2)^2$.

Question 15

Can you think of a function X, which is a function of t, such that when you differentiate that function and add the function you get $3t^2$.

Question 16

Can you think of a function X, which is a function of t, such that when you differentiate that function and then multiply the derivative by itself (that is square it) and then add the derivative of the function you get 13.

Question 17

Can you think of a function y, which is a function of x, such that when you differentiate that function and then multiply the derivative by itself (that is square it) and then multiply the square of the derivative by the function and then add $\sin(x)$ you get 0.

Question 18

Can you think of a function Z, which is a function of Y, such that when you differentiate that function and then multiply the derivative by the function and then add 3 times the function you get Y^2.

Question 19

Can you think of a function y, which is a function of x, such that when you multiply the derivative of that function by 2, add 3 times the function and then add $\sin(x)$ you get 0.

Question 20

Can you think of a function X, which is a function of t, such that when you differentiate that function and then multiply the derivative by itself (that is square it) and then add the function times $\sin(t)$ you get t^2.

5.7 Answers to Exercises 2.2.5

Question 1 (e).

Question 2 (c).

Question 3 (d).

Question 4 (f).

Question 5 (a) represents (iii), (b) represents (v), (c) represents (i), (d) represents (ii), (e) represents (iv).

5.8 Answers to Exercises 2.3.2

Question 1 (a), (b), (c), (d).

Question 2 (a), (b).

Question 3 (c), (e). The answer (e) qualifies as a solution as; if d is a constant then $2d$ will also be a constant. The constant d in part (d) corresponds to $c/2$ in the general solution of part (c).

5.9 Answers to Exercises 2.3.4

Question 1 (a) (i), (b) (iii), (c) (ii), (d) (v), (e) (vi), (f) (ii).

Question 2 (a) (vi), (b) (i), (c) (ii), (d) (iv), (e) (iii), (f) (i).

Question 3 (a) matches none of the descriptions, (b) (ii), (c) matches none of the descriptions, (d) (i), (e) (iii), (f) (ii).

Question 4 (a) matches none of the descriptions, (b) (i), (c) (ii), (d) matches none of the descriptions, (e) (iii), (f) (ii).

5.10 Answers to Exercises 3.1.1

Question 1 (b), (d).

Question 2 (d).

Question 3 (a).

Question 4 (e).

Question 5 (c), (d).

Question 6 (d).

Question 7 (a), (b), (c).

Question 8 (c).

Question 9 (d).

Question 10 (a), (b), (c), (d).

5.11 Answers to Exercises 3.3.1

Question 1 $y(x) = \frac{3}{4}x^4 + c.$

Question 2 $y(x) = 2x + c.$

Question 3 $y(x) = \frac{5}{7}x^7 - 4x^4 + c.$

Question 4 $y(x) = \frac{\pi}{3}x^3 + c.$

Question 5 $y(x) = \sin(x) + c.$

Question 6 $y(x) = \frac{1}{5}e^{5x} + c.$

Question 7 $y(x) = x^2 + \sin(x) + c.$

Question 8 $y(x) = \frac{1}{3}\sin(3x) + c.$

Question 9 $y(x) = \sin(x^2) + c.$

Question 10 $y(x) = e^{x^2} + c.$

Question 11 $y(x) = 5\ln(x) - \cos(3x^5) + c.$

5.12 Answers to Exercises 3.4.1

Question 1

$$\begin{aligned}
\text{Left hand side of d.e. is } \quad \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{3}{4}x^4\right) \\
&= \frac{3}{4} \times 4x^3 \\
&= 3x^3 \\
&= \text{Right hand side of d.e.}
\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 2

$$\begin{aligned}
\text{Left hand side of d.e. is } \quad \frac{d}{dx}y &= \frac{d}{dx}(2x + c) \\
&= 2 + \frac{d}{dx}c \\
&= 2 \\
&= \text{Right hand side of d.e.}
\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 3

$$\begin{aligned}
\text{Left hand side of d.e. is } \quad \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{5}{7}x^7 - \frac{16}{4}x^4\right) \\
&= 5x^6 - 16x^3 \\
&= \text{Right hand side of d.e.}
\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 4

$$\begin{aligned}
\text{Left hand side of d.e. is } \quad \frac{d}{dx}y &= \frac{d}{dx}\left(\pi\frac{x^3}{3} + c\right) \\
&= \pi x^2 + \frac{d}{dx}c \\
&= \pi x^2 \\
&= \text{Right hand side of d.e.}
\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 5

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}(\sin(x)+3)\\ &= +\cos(x)\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 6

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right)\\ &= e^{5x}\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 7

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{x^2}{2}+\sin(x)\right)\\ &= x+\cos(x)\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 8

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(\frac{1}{3}\sin(3x)\right)\\ &= \frac{1}{3}\times\cos(3x)\times 3\\ &= \cos(3x)\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 9

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(\sin(x^2)-\pi\right)\\ &= \cos(x^2)\times 2x+\frac{d}{dx}\pi\\ &= \cos(x^2)\times 2x\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 10

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(e^{x^2}+c\right)\\ &= e^{x^2}\times 2x+\frac{d}{dx}c\\ &= e^{x^2}\times 2x\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

Question 11

$$\begin{aligned}\text{Left hand side of d.e. is } \frac{d}{dx}y &= \frac{d}{dx}\left(x^5\cos(15x^4)\right)\\ &= 5x^4\cos(15x^4)-x^5\sin(15x^4)\times 15\times 4x^3\\ &= 5x^4\cos(15x^4)-60x^8\sin(15x^4)\\ &= \text{Right hand side of d.e.}\end{aligned}$$

Which verifies that the function is a solution of the differential equation.

5.13 Answers to Exercises 3.3.1

Question 1 $y(x) = \frac{3}{4}x^4 + c$.

Question 2 $y(x) = 2x + c$.

Question 3 $y(x) = \frac{5}{7}x^7 - 4x^4 + c$.

Question 4 $y(x) = \frac{\pi}{3}x^3 + c$.

Question 5 $y(x) = \sin(x) + c$.

Question 6 $y(x) = \frac{1}{5}e^{5x} + c$.

Question 7 $y(x) = x^2 + \sin(x) + c$.

Question 8 $y(x) = \frac{1}{3}\sin(3x) + c$.

Question 9 $y(x) = \sin(x^2) + c$.

Question 10 $y(x) = e^{x^2} + c$.

Question 11 $y(x) = 5\ln(x) - \cos(3x^5) + c$.

5.14 Answers to Exercises 3.6.2

Question 1 $y(x) = e^x$, $y(x) = 5e^x$, $y(x) = -153\pi e^x$, as examples. There are many more.

Question 2 Can you think of a function X, which is a function of t, such that when you differentiate that function you get the same function you first thought of.

Question 3 $X(t) = e^t$, $X(t) = 5e^t$, $X(t) = -153\pi e^t$, as examples.

Question 4 $X(t) = Ae^t$, for A constant.

Question 6 Particular solutions $Z(y) = e^y$, $Z(y) = -21e^y$, $Z(y) = -(21e^{\pi-e} + \sin(e^2) - 15\pi^{3e-4})e^y$; general solution $Z(y) = Ae^y$.

5.15 Answers to Exercises 3.6.4

Question 1 (b), (c), (d).

Question 2 (c), (d).

Question 3 (a) (iv), (b) (v), (c) (vi), (d) (iii), (e) (i), (f) (ii).

Question 4 (a) (v), (b) (ii), (c) (iii), (d) (i), (e) (vi), (f) (iv).

5.16 Answers to Exercises 4.1.1

Question 1

(a) When $t = 0$ $H = 15$.

(b) When $t = 1$ $H = 13.5$.

(c) When $t = 0$ $H = 24$.

(d) When $t = 48$ $H = 12.7$.

(e) When $t = 0$ $H = 1.55$.

(f) When $t = 72$ $H = 2.44$.

(g) When $t = 24$ $H = 7$.

(h) When $t = 0$ $H = 15.3$.

Question 2

(a) When $t = 0$ $M = 150$.

(b) When $t = 150$ $M = 45$.

(c) When $t = 0$ $M = 250$.

(d) When $t = 2880$ $M = 17$.

(e) When $\dfrac{dM}{dt} = 12$ $M = 175$.

(f) When $t = 300$ $M'(t) = 2$.

(g) When $t = 60$ $M'(t) = 12.5$.

(h) When $t = 0$ $M = 15.3$.

(i) When $t = 60$ $M' = 3$.

(j) When $t = 360$ $\dfrac{dM}{dt} = 3.5$.

(k) When $t = 60$ $H = 145$.

Question 3

(a) When $t = 0$ $P = 2.34$.

(b) When $t = 3.5$ $P = 1.324$.

(c) When $t = 0$ $P'(t) = 2$.

(d) When $t = 48$ $P' = -0.524$.

(e) $R = 0.5$.

(f) When $t = 582$ $P = 0$.

(g) When $t = 24$ $P = 22.564$.

(h) When $t = 24$ $P = 1000$. Note if P is given as 1000 this must already be in units of 1000s. For $P = 1000$ there are $1000,000$ fish!

(i) When $t = 0$ $\dfrac{dP}{dt} = 2.0$.

(j) When $t = 24$ $P' = 2.0$.

(k) When $t = 150$ $P = 0$. Or this is saying after 150 hours there are no fish left.