Mathematics Learning Centre

The University of Sydney

Polynomials

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1 Polynomials

Many of the functions we will examine will be polynomials. In this Chapter we will study them in more detail.

Definition

A real *polynomial*, $P(x)$, of degree n is an expression of the form

$$P(x) = p_n x^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + p_0$$

where $p_n \neq 0$, p_0, p_1, $\cdots$, p_n are real and n is an integer ≥ 0.

All polynomials are defined for all real x and are continuous functions.

We are familiar with the quadratic polynomial, $Q(x) = ax^2 + bx + c$ where $a \neq 0$. This polynomial has degree 2.

The function $f(x) = \sqrt{x} + x$ is not a polynomial as it has a power which is not an integer ≥ 0 and so does not satisfy the definition.

1.1 Polynomial equations and their roots

If, for a polynomial $P(x)$, $P(k) = 0$ then we can say

1. $x = k$ is a root of the equation $P(x) = 0$.

2. $x = k$ is a zero of $P(x)$.

3. k is an x-intercept of the graph of $P(x)$.

1.1.1 Zeros of the quadratic polynomial

The quadratic polynomial equation $Q(x) = ax^2 + bx + c = 0$ has two roots that may be:

1. real (rational or irrational) and distinct,

2. real (rational or irrational) and equal,

3. complex (not real).

We can determine which one of these we have if we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, there will be two real distinct roots.

If $b^2 - 4ac = 0$, there will be two real equal roots.

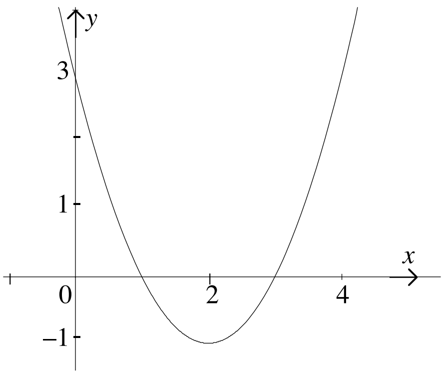
If $b^2 - 4ac < 0$, there will be complex roots.

We will illustrate all of these cases with examples, and will show the relationship between the nature and number of zeros of $Q(x)$ and the x-intercepts (if any) on the graph.

1. Let $Q(x) = x^2 - 4x + 3$.
We find the zeros of $Q(x)$ by solving the equation $Q(x) = 0$.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ \text{Therefore } x &= 1 \text{ or } 3. \end{aligned}$$

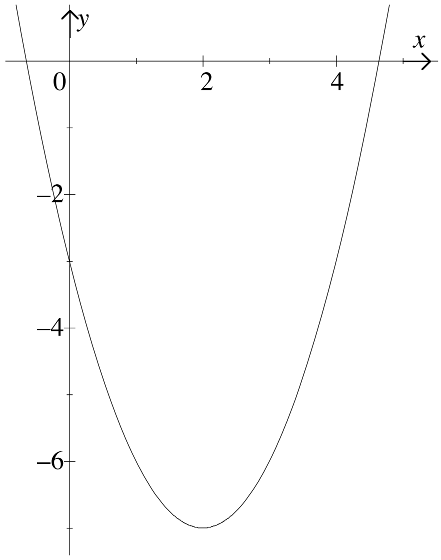
The roots are rational (hence real) and distinct.



2. Let $Q(x) = x^2 - 4x - 3$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x - 3 &= 0 \\ x &= \frac{4 \pm \sqrt{16+12}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{7}. \end{aligned}$$

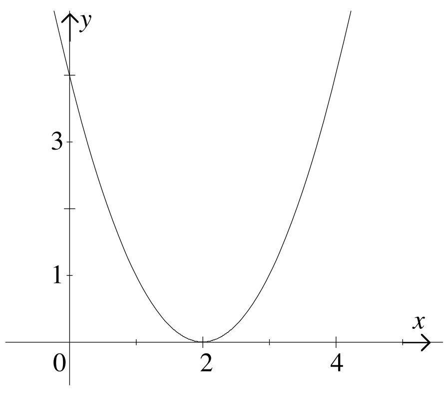
The roots are irrational (hence real) and distinct.



3. Let $Q(x) = x^2 - 4x + 4$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ \text{Therefore } x &= 2. \end{aligned}$$

The roots are rational (hence real) and equal. $Q(x) = 0$ has a repeated or double root at $x = 2$.

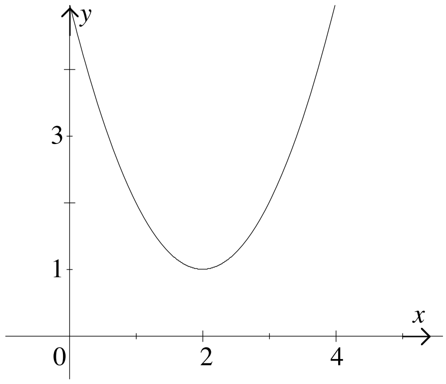


Notice that the graph turns at the double root $x = 2$.

4. Let $Q(x) = x^2 - 4x + 5$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 5 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 20}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{-4}. \end{aligned}$$

There are no real roots. In this case the roots are complex.



Notice that the graph does not intersect the x-axis. That is $Q(x) > 0$ for all real x. Therefore Q is positive definite.

We have given above four examples of quadratic polynomials to illustrate the relationship between the zeros of the polynomials and their graphs.

In particular we saw that:

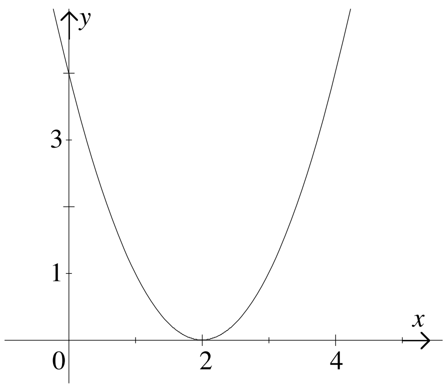
i. if the quadratic polynomial has two real distinct zeros, then the graph of the polynomial cuts the x-axis at two distinct points;

ii. if the quadratic polynomial has a real double (or repeated) zero, then the graph sits on the x-axis;

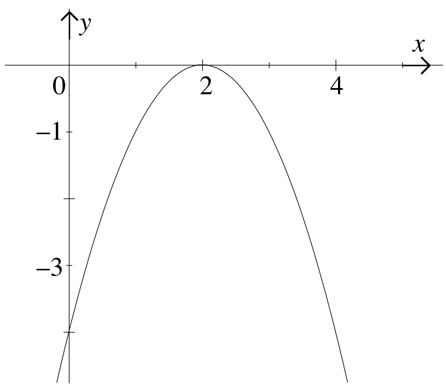
iii. if the quadratic polynomial has no real zeros, then the graph does not intersect the x-axis at all.

So far, we have only considered quadratic polynomials where the coefficient of the x^2 term is positive which gives us a graph which is *concave up*. If we consider polynomials $Q(x) = ax^2 + bx + c$ where $a < 0$ then we will have a graph which is *concave down*.

For example, the graph of $Q(x) = -(x^2 - 4x + 4)$ is the reflection in the x-axis of the graph of $Q(x) = x^2 - 4x + 4$.



The graph of $Q(x) = x^2 - 4x + 4$.



The graph of $Q(x) = -(x^2 - 4x + 4)$.

1.1.2 Zeros of cubic polynomials

A real cubic polynomial has an equation of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

where $a \neq 0$, a, b, c and d are real. It has 3 zeros which may be:

i. 3 real distinct zeros;

ii. 3 real zeros, all of which are equal (3 equal zeros);

iii. 3 real zeros, 2 of which are equal;

iv. 1 real zero and 2 complex zeros.

We will not discuss these here.

1.2 Factorising polynomials

So far for the most part, we have looked at polynomials which were already factorised. In this section we will look at methods which will help us factorise polynomials with degree > 2.

1.2.1 Dividing polynomials

Suppose we have two polynomials $P(x)$ and $A(x)$, with the degree of $P(x) \geq$ the degree of $A(x)$, and $P(x)$ is divided by $A(x)$. Then

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)},$$

where $Q(x)$ is a polynomial called the *quotient* and $R(x)$ is a polynomial called the *remainder*, with the degree of $R(x) <$ degree of $A(x)$.

We can rewrite this as

$$P(x) = A(x) \cdot Q(x) + R(x).$$

For example: If $P(x) = 2x^3 + 4x + 3$ and $A(x) = x - 2$, then $P(x)$ can be divided by $A(x)$ as follows:

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 2x^2 & +\;4x & +\;12 & \\ \cline{2-5}
x-2 & 2x^3 & +\;0x^2 & +\;4x & -\;3 \\
\multicolumn{1}{r}{} & 2x^3 & -\;4x^2 & & \\ \cline{2-5}
\multicolumn{1}{r}{} & & 4x^2 & +\;4x & -\;3 \\
\multicolumn{1}{r}{} & & 4x^2 & -\;8x & \\ \cline{3-5}
\multicolumn{1}{r}{} & & & 12x & -\;3 \\
\multicolumn{1}{r}{} & & & 12x & -\;24 \\ \cline{4-5}
\multicolumn{1}{r}{} & & & & 21
\end{array}
$$

The quotient is $2x^2 + 4x + 12$ and the remainder is 21. We have

$$\frac{2x^3 + 4x + 3}{x - 2} = 2x^2 + 4x + 12 + \frac{21}{x - 2}.$$

This can be written as

$$2x^3 + 4x - 3 = (x - 2)(2x^2 + 4x + 12) + 21.$$

Note that the degree of the "polynomial" 21 is 0.

1.2.2 The Remainder Theorem

If the polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.
Proof:
Following the above, we can write

$$f(x) = A(x) \cdot Q(x) + R(x),$$

where $A(x) = (x - a)$. Since the degree of $A(x)$ is 1, the degree of $R(x)$ is zero. That is, $R(x) = r$ where r is a constant.

$$\begin{aligned} f(x) &= (x-a)Q(x) + r \quad \text{where } r \text{ is a constant.} \\ f(a) &= 0 \cdot Q(a) + r \\ &= r \end{aligned}$$

So, if $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Example

Find the remainder when $P(x) = 3x^4 - x^3 + 30x - 1$ is divided by **a.** $x + 1$, **b.** $2x - 1$.
Solution
a. Using the Remainder Theorem:

$$\begin{aligned} \text{Remainder} &= P(-1) \\ &= 3 - (-1) - 30 - 1 \\ &= -27 \end{aligned}$$

b.

$$\begin{aligned} \text{Remainder} &= P(\frac{1}{2}) \\ &= 3(\frac{1}{2})^4 - (\frac{1}{2})^3 + 30(\frac{1}{2}) - 1 \\ &= \frac{3}{16} - \frac{1}{8} + 15 - 1 \\ &= 14\frac{1}{16} \end{aligned}$$

Example

When the polynomial $f(x)$ is divided by $x^2 - 4$, the remainder is $5x + 6$. What is the remainder when $f(x)$ is divided by $(x - 2)$?

Solution
Write $f(x) = (x^2 - 4) \cdot q(x) + (5x + 6)$. Then

$$\begin{aligned} \text{Remainder} &= f(2) \\ &= 0 \cdot q(2) + 16 \\ &= 16 \end{aligned}$$

A consequence of the Remainder Theorem is the Factor Theorem which we state below.

1.2.3 The Factor Theorem

If $x = a$ is a zero of $f(x)$, that is $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$ and $f(x)$ may be written as

$$f(x) = (x - a)q(x)$$

for some polynomial $q(x)$.

Also, if $(x - a)$ and $(x - b)$ are factors of $f(x)$ then $(x - a)(x - b)$ is a factor of $f(x)$ and

$$f(x) = (x - a)(x - b) \cdot Q(x)$$

for some polynomial $Q(x)$.

Another useful fact about zeros of polynomials is given below for a polynomial of degree 3.

1.2.4 The cubic polynomial revisited

If a (real) polynomial

$$P(x) = ax^3 + bx^2 + cx + d,$$

where $a \neq 0$, a, b, c and d are real, has exactly 3 real zeros α, β and γ, then

$$P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \qquad (1)$$

Furthermore, by expanding the right hand side of (1) and equating coefficients we get:

i.

$$\alpha + \beta + \gamma = -\frac{b}{a};$$

ii.

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a};$$

iii.

$$\alpha\beta\gamma = -\frac{d}{a}.$$

This result can be extended for polynomials of degree n.

Example

Let $f(x) = 4x^3 - 8x^2 - x + 2$

a. Factorise $f(x)$.

b. Sketch the graph of $y = f(x)$.

c. Solve $f(x) \geq 0$.

Solution

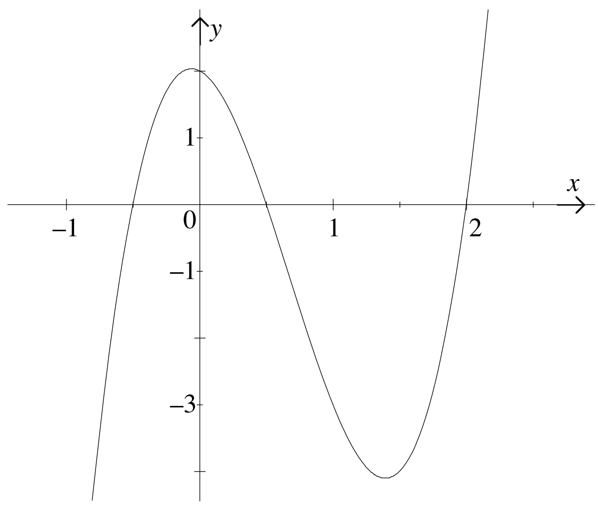
a. Consider the factors of the constant term, 2. We check to see if ± 1 and ± 2 are solutions of the equation $f(x) = 0$ by substitution. Since $f(2) = 0$, we know that $(x - 2)$ is a factor of $f(x)$. We use long division to determine the quotient.

$$\begin{array}{r|l} & 4x^2 - 1 \\ \hline x - 2 & 4x^3 - 8x^2 - x + 2 \\ & 4x^3 - 8x^2 \\ \hline & \qquad\quad - x + 2 \\ & \qquad\quad - x + 2 \\ \hline \end{array}$$

So,

$$\begin{aligned} f(x) &= (x-2)(4x^2-1) \\ &= (x-2)(2x-1)(2x+1) \end{aligned}$$

b.



The graph of $f(x) = 4x^3 - 8x^2 - x + 2$.

c. $f(x) \geq 0$ when $-\frac{1}{2} \leq x \leq \frac{1}{2}$ or $x \geq 2$.

Example

Show that $(x - 2)$ and $(x - 3)$ are factors of $P(x) = x^3 - 19x + 30$, and hence solve $x^3 - 19x + 30 = 0$.

Solution

$P(2) = 8 - 38 + 30 = 0$ and $P(3) = 27 - 57 + 30 = 0$ so $(x - 2)$ and $(x - 3)$ are both factors of $P(x)$ and $(x - 2)(x - 3) = x^2 - 5x + 6$ is also a factor of $P(x)$. Long division of $P(x)$ by $x^2 - 5x + 6$ gives a quotient of $(x + 5)$.

So,

$$P(x) = x^3 - 19x + 30 = (x-2)(x-3)(x+5).$$

Solving $P(x) = 0$ we get $(x-2)(x-3)(x+5) = 0$.

That is, $x = 2$ or $x = 3$ or $x = -5$.

Instead of using long division we could have used the facts that

i. the polynomial cannot have more than three real zeros;

ii. the product of the zeros must be equal to -30.

Let α be the unknown root.

Then $2 \cdot 3 \cdot \alpha = -30$, so that $\alpha = -5$. Therefore the solution of $P(x) = x^3 - 19x + 30 = 0$ is $x = 2$ or $x = 3$ or $x = -5$.

1.3 Exercises

1. When the polynomial $P(x)$ is divided by $(x-a)(x-b)$ the quotient is $Q(x)$ and the remainder is $R(x)$.

- **a.** Explain why $R(x)$ is of the form $mx + c$ where m and c are constants.
- **b.** When a polynomial is divided by $(x-2)$ and $(x-3)$, the remainders are 4 and 9 respectively. Find the remainder when the polynomial is divided by $x^2 - 5x + 6$.
- **c.** When $P(x)$ is divided by $(x-a)$ the remainder is a^2. Also, $P(b) = b^2$. Find $R(x)$ when $P(x)$ is divided by $(x-a)(x-b)$.

2.
- **a.** Divide the polynomial $f(x) = 2x^4 + 13x^3 + 18x^2 + x - 4$ by $g(x) = x^2 + 5x + 2$. Hence write $f(x) = g(x)q(x) + r(x)$ where $q(x)$ and $r(x)$ are polynomials.
- **b.** Show that $f(x)$ and $g(x)$ have no common zeros. (Hint: Assume that α is a common zero and show by contradiction that α does not exist.)

3. For the following polynomials,

- **i.** factorise
- **ii.** solve $P(x) = 0$.

- **a.** $P(x) = x^3 - x^2 - 10x - 8$
- **b.** $P(x) = x^3 - x^2 - 16x - 20$
- **c.** $P(x) = x^3 + 4x^2 - 8$
- **d.** $P(x) = x^3 - x^2 + x - 6$
- **e.** $P(x) = 2x^3 - 3x^2 - 11x + 6$

1.4 Solutions

1.
- **a.** Since $A(x) = (x-a)(x-b)$ is a polynomial of degree 2, the remainder $R(x)$ must be a polynomial of degree < 2. So, $R(x)$ is a polynomial of degree ≤ 1. That is, $R(x) = mx + c$ where m and c are constants. Note that if $m = 0$ the remainder is a constant.

b. Let $P(x) = (x^2 - 5x + 6)Q(x) + (mx + c) = (x - 2)(x - 3)Q(x) + (mx + c)$.
Then

$$\begin{aligned} P(2) &= (0)(-1)Q(2) + (2m + c) \\ &= 2m + c \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} P(3) &= (1)(0)Q(3) + (3m + c) \\ &= 3m + c \\ &= 9 \end{aligned}$$

Solving simultaneously we get that $m = 5$ and $c = -6$. So, the remainder is $R(x) = 5x - 6$.

c. Let $P(x) = (x - a)(x - b)Q(x) + (mx + c)$.
Then

$$\begin{aligned} P(a) &= (0)(a - b)Q(a) + (ma + c) \\ &= am + c \\ &= a^2 \end{aligned}$$

and

$$\begin{aligned} P(b) &= (b - a)(0)Q(b) + (mb + c) \\ &= bm + c \\ &= b^2 \end{aligned}$$

Solving simultaneously we get that $m = a + b$ and $c = -ab$ provided $a \neq b$.
So, $R(x) = (a + b)x - ab$.

2. a.

$$2x^4 + 13x^3 + 18x^2 + x - 4 = (x^2 + 5x + 2)(2x^2 + 3x - 1) - 2$$

b. Let α be a common zero of $f(x)$ and $g(x)$. That is, $f(\alpha) = 0$ and $g(\alpha) = 0$.
Then since $f(x) = g(x)q(x) + r(x)$ we have

$$\begin{aligned} f(\alpha) &= g(\alpha)q(\alpha) + r(\alpha) \\ &= (0)q(\alpha) + r(\alpha) \qquad \text{since } g(\alpha) = 0 \\ &= r(\alpha) \\ &= 0 \qquad \text{since } f(\alpha) = 0 \end{aligned}$$

But, from part **b.** $r(x) = -2$ for all values of x, so we have a contradiction.
Therefore, $f(x)$ and $g(x)$ do not have a common zero.
This is an example of a proof by contradiction.

3. a. i. $P(x) = x^3 - x^2 - 10x - 8 = (x + 1)(x + 2)(x - 4)$

ii. $x = -1$, $x = -2$ and $x = 4$ are solutions of $P(x) = 0$.

b. i. $P(x) = x^3 - x^2 - 16x - 20 = (x+2)^2(x-5)$.
ii. $x = -2$ and $x = 5$ are solutions of $P(x) = 0$. $x = -2$ is a double root.

c. i. $P(x) = x^3+4x^2-8 = (x+2)(x^2+2x-4) = (x+2)(x-(-1+\sqrt{5}))(x-(-1-\sqrt{5}))$
ii. $x = -2$, $x = -1+\sqrt{5}$ and $x = -1-\sqrt{5}$ are solutions of $P(x) = 0$.

d. i. $P(x) = x^3 - x^2 + x - 6 = (x-2)(x^2+x+3)$. $x^2+x+3 = 0$ has no real solutions.
ii. $x = 2$ is the only real solution of $P(x) = 0$.

e. i. $P(x) = 2x^3 - 3x^2 - 11 + 6 = (x+2)(x-3)(2x-1)$.
ii. $x = -2$, $x = \frac{1}{2}$ and $x = 3$ are solutions of $P(x) = 0$.