Mathematics Learning Centre

The University of Sydney

Solving simultaneous equations

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1 Simultaneous linear equations

We will introduce two methods for solving simultaneous linear equation with two variables. It is worth understanding and practising both methods so that you can select the method that is easier in any given situation.

1.1 The substitution method

Let's suppose we have a pair of linear equations $y = 2x + 2$ and $y = -x + 5$. Their graphs are illustrated in Figure 1.

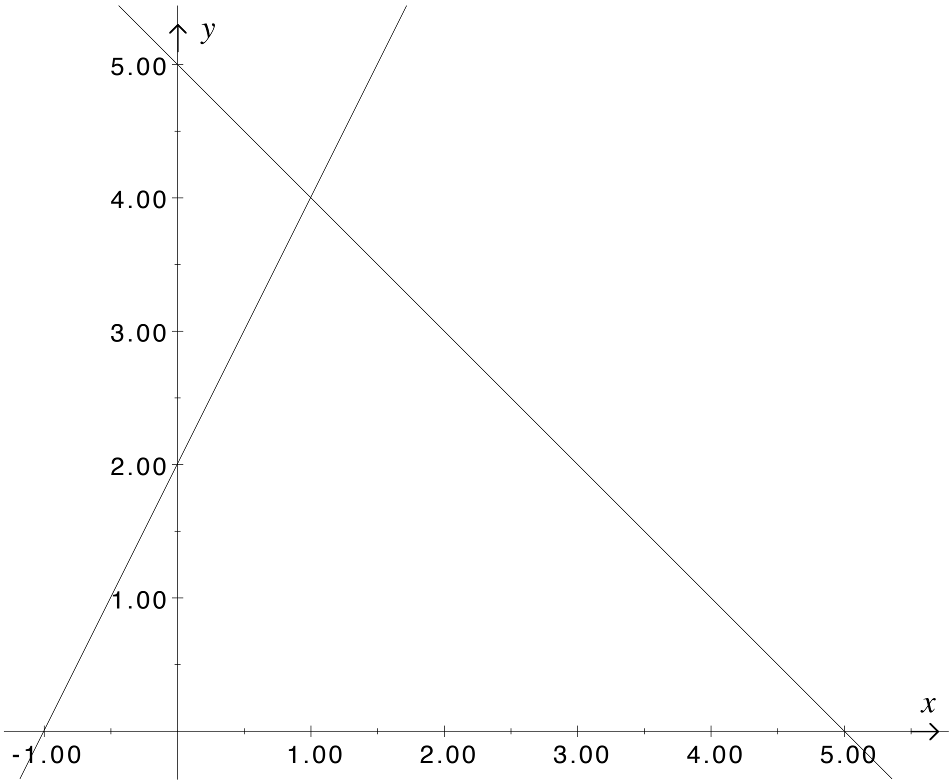

Figure 1: Straight line graphs of $y = 2x + 2$ and $y = -x + 5$.

Figure 1 shows a point of intersection. We want to be able to solve the equations algebraically to find this point of intersection, which looks to be at (1,4).

Since both equations are in the form $y = f(x)$ we can equate the right hand sides of the equations and solve for x.

$$\begin{aligned} 2x + 2 &= -x + 5 \\ 3x &= 3 \\ x &= 1. \end{aligned}$$

We can now substitute $x = 1$ into either equation to find y: $y = 2(1) + 2 = 4$.

So, we confirm that the point of intersection is (1,4).

This is the principle of solving simultaneous linear equations using the substitution method. In this case it helped that both equations were in the form $y = \ldots$ as this allowed us to equate and solve for x straight away.

However our equations could be in the form $2x - y + 2 = 0$ and $x + y - 5 = 0$. If this is the case we need to write one of them in the form $y = \ldots$ or $x = \ldots$ and then substitute the result in the other equation. These steps are carried out below.

$$2x - y + 2 = 0 \tag{1}$$

$$x + y - 5 = 0 \tag{2}$$

Rewrite (1) as follows:

$$y = 2x + 2. \tag{3}$$

Substitute (3) in equation (2):

$$x + (2x + 2) - 5 = 3x - 3 = 0.$$

So, $x = 1$ as before.

We can now substitute $x = 1$ into either equation (1) or equation (2) and solve for y.

$$2(1) - y + 2 = 4 - y = 0.$$

So $y = 4$.

We could check our answer by substituting these values into equation (2): $x + y - 5 = 1 + 4 - 5 = 0$ as required.

Example

Solve the pair of simultaneous equations: $3x + y = 13$ and $x + 2y = 1$.

Solution

$$3x + y = 13 \tag{4}$$

and

$$x + 2y = 1. \tag{5}$$

Rewrite equation (4) as:

$$y = 13 - 3x. \tag{6}$$

Substitute (6) in (5):

$$x + 2(13 - 3x) = -5x + 26 = 1.$$

So, $-5x = -25$, ie $x = 5$.

Now substitute $x = 5$ in equation (4):

$$3(5) + y = 15 + y = 13.$$

So, $y = -2$.

Check by substituting $x = 5$ and $y = -2$ into (5): $5 + 2(-2) = 1$ as required.

1.2 Solving simultaneous equations by the elimination method

Suppose we have a pair of simultaneous equations, $2x - y = -2$ and $x + y = 5$. We can solve these equations by taking the sum of the left hand sides and equating it to the sum of the right hand sides as follows:

$$2x - y + (x + y) = 3x = 3.$$

So, $x = 1$.

We were able to eliminate y by doing this as y in both equations had coefficient 1, and $y - y = 0$.

We can modify this method to work for all pairs of equations:

- select which variable to eliminate
- multiply one or both equations by a constant so that the coefficient of the 'elimination variable' in each equation is the same
- add or subtract to eliminate the chosen variable.

Let's illustrate this method with an example.

Example Solve the pair of equations $5x + 2y = 10$ and $4x + 3y = 15$.

Solution

$$5x + 2y = 10 \tag{7}$$

and

$$4x + 3y = 15. \tag{8}$$

Choosing to eliminate y, multiply equation (7) by 3 and equation (8) by 2 to get:

$$15x + 6y = 30 \tag{9}$$

and

$$8x + 6y = 30. \tag{10}$$

As the coefficients and the sign of y in each equation are the same, we subtract (10) from (9) to get:

$$15x - 8x = 7x = 30 - 30 = 0.$$

So, $x = 0$.

Substituting $x = 0$ into equation (8) we get $4(0) + 3y = 15$ ie $y = 5$.

Check by substituting into equation (7): $5(0) + 2(5) = 10$ as required.

Frequently the elimination method for solving pairs of simultaneous linear equations is slightly easier than the substitution method but you should know both methods.

1.3 Systems of equations with more than two variables

There may be situations where we have more than two equations and two variables. There are sophisticated methods to solve these systems, eg matrix algebra.

However, if we have three equations and three variables, we can adapt the methods we have here.

For example, suppose we have the following equations:

$$2x + y + \lambda = 0 \tag{11}$$

$$x + 4y - 3\lambda = 0 \tag{12}$$

and

$$x - 3y - 16 = 0 \tag{13}$$

Here, we can eliminate λ from equations (11) and (12) as follows:

Multiply (11) by 3

$$6x + 3y + 3\lambda = 0 \tag{14}$$

and add (14) and (12) to get

$$7x + 7y = 0. \tag{15}$$

This gives us $y = -x$ which we can substitute in (13):

$$x - 3(-x) - 16 = 4x - 16 = 0.$$

So that $x = 4$, $y = -4$ and $2(4) + (-4) + \lambda = 0$ ie $\lambda = -4$.

Note that we used both the substitution and elimination method here.

1.4 Exercises

Solve the following pairs of simultaneous equations using either the substitution method or the elimination method (but practice both).

1. $y = -3x + 2$ and $y = 2x - 8$

2. $2y - x = 4$ and $2x - 3y = 2$

3. $x + y = 7$ and $2x - y = 5$

4. $a + b - 12 = 0$ and $2a + b - 6 = 0$

5. $5x + 3y = -6$ and $6x - 2y = 32$

6. $9m - 7n = 3$ and $3m - 2n = -1$

7. $2x - 2\lambda = 0$, $2y - 3\lambda = 0$ and $2x - 3y - 5 = 0$

8. $8q + p + \lambda = 0$, $q + 2p + \lambda = 0$ and $q + p - 1600 = 0$

Answers to exercises

1. $x = 2$ and $y = -4$

2. $x = 16$ and $y = 10$

3. $x = 4$ and $y = 3$

4. $a = -6$ and $b = 18$

5. $x = 3$ and $y = -7$

6. $m = -\frac{13}{3}$ and $n = -6$

7. $x = -2$, $y = -3$ and $\lambda = -2$

8. $q = 200$, $p = 1400$ and $\lambda = -3000$