Solving systems of linear equations using the inverse

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Systems of linear equations

Matrix algebra can be used to represent systems of linear equations. Consider the following system:

$$\begin{aligned} 3x - 2y + 4z &= -5 \\ y + 2z &= 0 \\ x + y + z &= 4. \end{aligned}$$

We can write this system as a matrix equation.

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$

then $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$, ie $AX = C$.

Systems of linear equations

$$\underset{\substack{\uparrow \\ A}}{\begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ X}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \underset{=}{=} \underset{\substack{\uparrow \\ C}}{\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}}$$

The matrix A is made up of the coefficients of the system.

The left hand side of the matrix equation, AX, is a 3×1 column matrix, and when multiplied out gives:

$$\begin{bmatrix} 3x + (-2)y + 4z \\ 0x + 1y + 2z \\ 1x + 1y + 1z \end{bmatrix} = \begin{bmatrix} 3x - 2y + 4z \\ y + 2z \\ x + y + z \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}.$$

Equating rows, we get back our system of equations again.

Solving the matrix equation

If A is a square matrix and has an inverse, A^{-1}, then we can solve the system of equations as follows:

$$\begin{array}{rcll} AX & = & C & \\ A^{-1}(AX) & = & A^{-1}C & \text{multiplying on the left by } A^{-1} \\ (A^{-1}A)(X) & = & A^{-1}C & \text{using associativity} \\ IX & = & A^{-1}C & A^{-1}A = I \\ X & = & A^{-1}C & \end{array}$$

Provided we have A^{-1} we can solve any system of n linear equations with n unknowns in this manner; the difficulty is finding A^{-1} if it exists.

We will see how this works for a simple example next.

Example

Solve the following system of equations:

$$\begin{aligned} 3x + y &= 13 \\ x + 2y &= 1. \end{aligned}$$

We can write this as $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix}$, ie $AX = C$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

$|A| = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 3(2) - 1(1) = 5$ so A^{-1} exists, and

$A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$.

Example continued

Recall, that if $AX = C$ and A^{-1} exists then $X = A^{-1}C$.

So for our example $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 13 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{25}{5} \\ -\frac{10}{5} \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -2 \end{bmatrix} \end{aligned}$$

so $x = 5$, and $y = -2$.