Mathematics Learning Centre

The University of Sydney

# Working with quadratic and exponential graphs

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# 1 Working with graphs

The quadratic, exponential and logarithmic functions frequently occur in economics or statistics. We will look at each in turn with the aim of giving you some understanding of these functions and their graphical representation.

## 1.1 Quadratic functions

A quadratic function has an equation of the form $y = ax^2 + bx + c$ where $a \neq 0$, $b$ and $c$ are constants.

Here are two scenarios where you may come across a quadratic function.

- A firm's revenue, $R$, is given as a function of output, $q$, by the formula,

$$R = 200q - 2q^2.$$

- The flow-density relationship of traffic flow on a highway can be described by the expression:

$$q = u_f k - \frac{u_f}{k_j} k^2.$$

  Here $q$ is given as a quadratic function of $k$, where $u_f$ and $k_j$ are constant.

The graph of a quadratic is always a parabola and looks like Figure 1 or Figure 2.

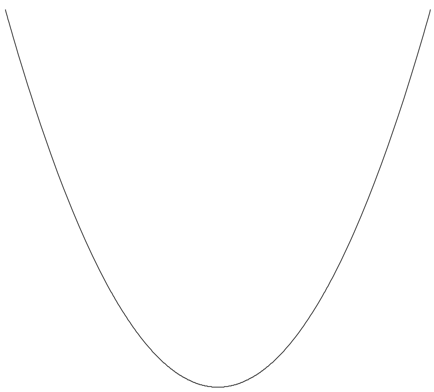


Fig. 1

or

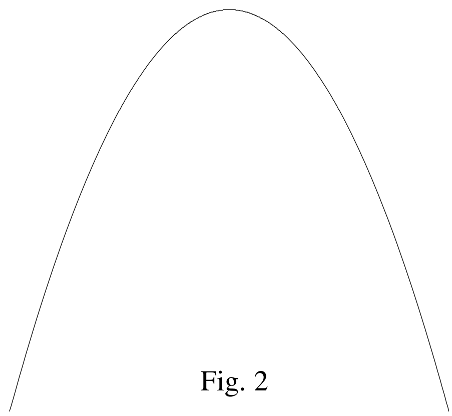


Fig. 2

The coefficient of the squared term determines whether your graph is like Figure 1 or Figure 2.

For the general polynomial $y = ax^2 + bx + c$, if $a > 0$ we get a parabola like that in Figure 1, while if $a < 0$, we get a parabola like that in Figure 2.

So, $R = 200q - 2q^2$ will look like the parabola in Figure 2, as the coefficient of the $q^2$ term is negative.

Note that $R$ has a maximum value.

For a general quadratic with equation $y = ax^2 + bx + c$, there is a maximum or a minimum value when $x = -\dfrac{b}{2a}$.

For $R = 200q - 2q^2$ will have a maximum at $q = -\dfrac{200}{2(-2)} = 50$.

Note also that the parabolas in Figures 1 and 2 are symmetric about the vertical line through the maximum or minimum.

To graph our quadratic we now need information that orients it on the coordinate axes.

One thing we can do is determine the coordinates of the maximum or minimum.

For the general polynomial, $y = ax^2 + bx + c$ we know this maximum or minimum occurs when $x = -\dfrac{b}{2a}$. To find the $y$ coordinate of this maximum or minimum we substitute this value into $y = ax^2 + bx + c$:

$$y = a(-\frac{b}{2a})^2 + b(-\frac{b}{2a}) + c = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c.$$

This gives us the coordinates of the maximum or minimum.

For our example $R = 200q - 2q^2$, there is a maximum at $q = 50$ so the maximum value of $R$ is

$$R = 200(50) - 2(50)^2 = 10000 - 5000 = 5000.$$

We now need either one or both intercepts of the parabola with the coordinate axes.

If $x = 0$ this is easy to find for $y = ax^2 + bx + c$, as when $x = 0$, $y = c$.

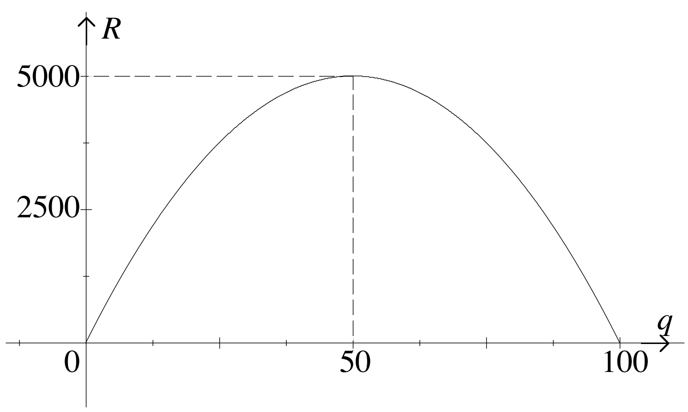
If we want the points, if any, where the parabola $y = ax^2 + bx + c$ crosses the $x$ axis, we set $y = 0$ and solve $ax^2 + bx + c = 0$ by factorising if we can, or by using the quadratic formula if required.

For our example, $R = 200q - 2q^2$, when $q = 0$, $R = 0$.

If $R = 0$ then $200q - 2q^2 = 2q(100 - q) = 0$ so, $q = 0$ or $q = 100$.

This tells us that the parabola crosses the $x$ axis at $(0, 0)$ and $(100, 0)$.

We now have sufficient information to sketch the graph of $R = 200q - 2q^2$.



**Example**

Sketch the graph of $q = u_f k - \frac{u_f}{k_j} k^2$ if $u_f$ and $k_j$ are both positive numbers.

**Solution**

$q = u_f k - \frac{u_f}{k_j} k^2$ will look like Figure 2 since $u_f$ and $k_j$ are both positive. Note that $q$ has a maximum value.

$q = u_f k - \frac{u_f}{k_j} k^2$ will have a maximum value when $k = -\frac{u_f}{2(-\frac{u_f}{k_j})} = \frac{k_j}{2}$.

$q = u_f k - \frac{u_f}{k_j} k^2$ has a maximum at $k = \frac{k_j}{2}$ so the maximum value for $q$ is

$$q = u_f(\frac{k_j}{2}) - \frac{u_f}{k_j}(\frac{k_j}{2})^2 = \frac{2u_f k_j - u_f k_j}{4} = \frac{u_f k_j}{4}.$$

$q = u_f k - \frac{u_f}{k_j} k^2$, when $k = 0$, $q = 0$.

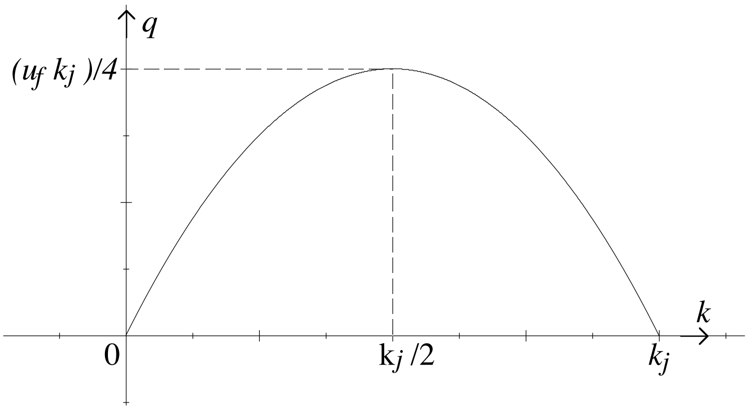
If $q = 0$ then $u_f k - \frac{u_f}{k_j} k^2 = 0$ so,

$$u_f k(1 - \frac{k}{k_j}) = 0.$$

That is when $k = 0$ or $k = k_j$.

So, the parabola crosses the $k$ axis at $k = 0$ and $k = k_j$.

We now have sufficient information to sketch the graph of $q = u_f k - \frac{u_f}{k_j} k^2$.



# 2 Exponential Functions

There is a class of functions called the exponential functions that are important in economics and statistics. For example, a Poisson probability distribution function is given by:

$$P(X) = \frac{e^{-\lambda}\lambda^X}{X!}.$$

In this example, $e^{-\lambda}$ is an exponential function, as its variable, $\lambda$, is in the power or exponent.

## 2.1 The functions $y = 2^x$ and $y = 2^{-x}$

The easiest function of this type to graph is the function $y = 2^x$ and we grph this function in Figure 1.

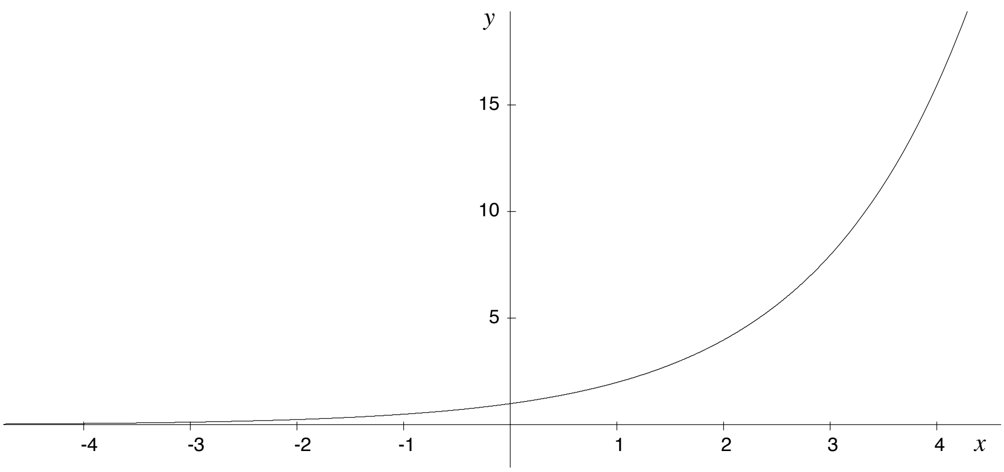



Figure 1: Graph of the function $f(x) = 2^x$

You should be aware of several important features of this graph.

The function $f(x) = 2^x$ is always positive (the graph of the function never cuts the $x$-axis), although the value of the function gets very close to zero for values of $x$ very large negative (ie a long way to the left along the $x$-axis). For example, when $x = -5$ we have $2^x = 0.03125$.

The function $2^x$ increases very rapidly for large values of $x$.

From the rules of exponents,

$$2^{x+1} = 2 \times 2^x.$$

In words, the value of $2^x$ doubles if $x$ is increased by 1.

The graph of $y = 2^x$ intercepts the $y$-axis at $y = 1$. You should expect this because $2^0 = 1$.

Figure 2 displays the graph of the function $f(x) = 2^{-x}$. How is the graph of $y = 2^{-x}$ related to the graph of $y = 2^x$?

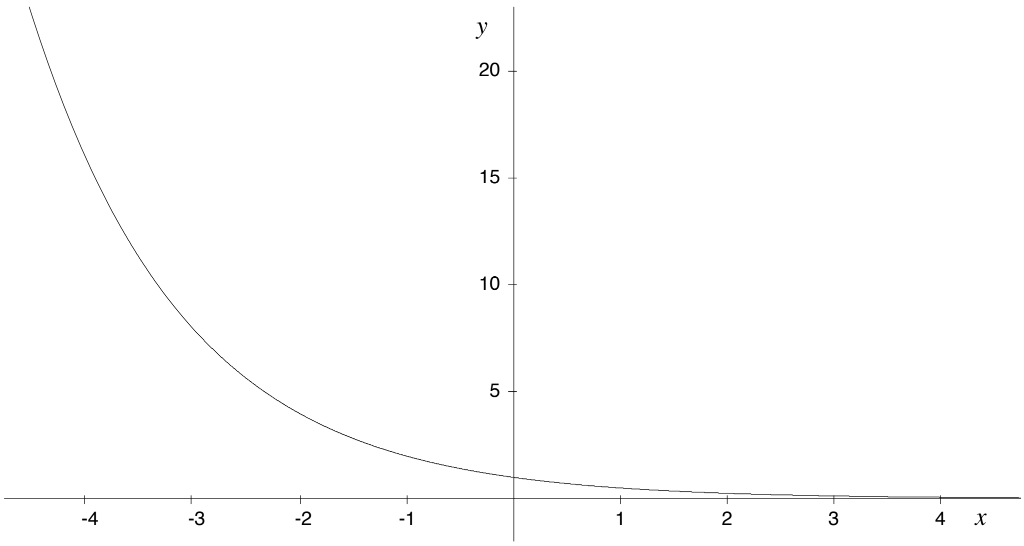



Figure 2: Graph of the function $y = 2^{-x}$

Well, if we set $x = 1$ then $2^{-x} = 2^{-1} = \frac{1}{2}$, which is the value which would have been obtained by setting $x = -1$ in the function $y = 2^x$.

In the same way we see that if we set $x = -7$ in the function $y = 2^{-x}$ then we obtain the same value as we would by setting $x = 7$ in the function $y = 2^x$.

Proceeding like this we see that *the graph of the function $y = 2^{-x}$ is the reflection in the* $y$-axis *of the graph of* $y = 2^x$. Compare Figure 1 with Figure 2.

It follows that

$$2^{-x} = (2^{-1})^x = (\frac{1}{2})^x.$$

The function $y = 2^{-x}$ is the same as the function $y = (\frac{1}{2})^x$, and so

$$2^{-(x+1)} = (\frac{1}{2})^{x+1} = \frac{1}{2} \times (\frac{1}{2})^x = (\frac{1}{2}) \times 2^{-x}.$$

In words, the value of the function $y = 2^{-x}$ is decreased by a factor of $\frac{1}{2}$ if $x$ is increased by 1.

## 2.2 The functions $y = e^x$ and $y = e^{-x}$

There is a number called $e$ which has a special importance in mathematics.

Like the number $\pi$, the number $e$ is an *irrational* number, which is equivalent to saying that it has a non-terminating, non-repeating decimal representation. In other words we can never write down exactly what $e$ is. To 5 decimal places it is equal to 2.71828, but this is just an approximation of the correct value.

Unless you really need to write down an approximate value for $e$ it is more convenient and accurate to leave the symbol $e$ in expressions involving this number. For example, it is preferable to write $2e$ rather than $2 \times 2.71828$ or 5.43656.

In mathematics the functions $e^x$ and $e^{-x}$ are particularly important. Because of this we have graphed them in Figure 3. You can see how similar these functions are to the exponential functions, $2^x$ and $2^{-x}$.

The function $y = e^x$ is often referred to as *the* exponential function, and is even given another special symbol, exp, so that $\exp(x) = e^x$ and $\exp(-x) = e^{-x}$.

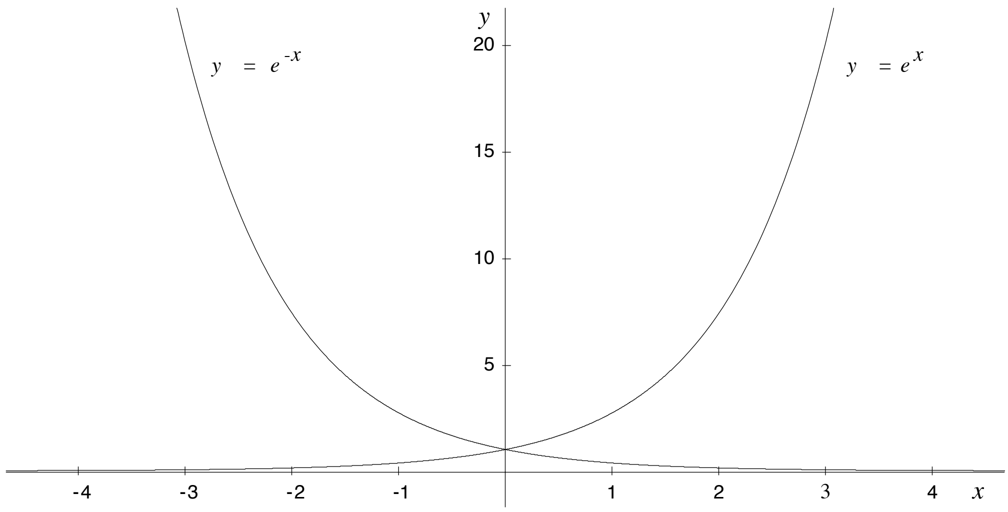



Figure 3: Graphs of $e^x$ and $e^{-x}$.

**Example**

The density function of the exponential distribution is given by:

$$y = \frac{1}{\lambda}e^{-\frac{x}{\lambda}} \qquad \text{where } x \geq 0.$$

Sketch a graph of this function when $\lambda = 0.5$.

**Solution**

When $\lambda = 0.5$, we have

$$y = \frac{1}{0.5}e^{-\frac{x}{0.5}} = 2e^{-2x}.$$

Therefore, when $x = 0$, $y = 2$. The graph of the density function is given in Figure 4.

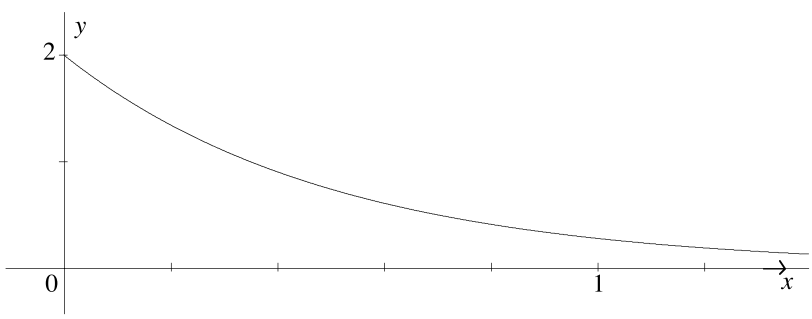



Figure 4: Graphs of $y = 2e^{-2x}$.