MATH 3963
NONLINEAR ODES WITH APPLICATIONS

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Contents

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1. A Mild Introduction

The purpose of this course is to answer the following questions:

- What is a differential equation?
- How can we best understand a given differential equation?

We can answer the first question right away, but answering the second question has occupied scientists/mathematicians since differential equations were originally formulated in the 17th century. The short answer to the first question is a *differential equation*[1] is a relationship between a function and its derivative. When the function depends only on a single variable, the differential equation is called an *ordinary differential equation* or an ODE. This is what this course is about. Compare this to a *partial differential equation*, which is a differential equation where the function(s) depends on more than one variable.

Usually our functions are functions of time, so we denote the independent variable by t. Also, typically we don't want to consider only (scalar valued) functions. We want to consider *vector valued functions*

$$\mathbf{y} : \mathbb{R} \to \mathbb{R}^d.$$

A differential equation is then an equation of the form

$$\mathbf{F}\left(t, \frac{d\mathbf{y}}{dt}, \frac{d^2\mathbf{y}}{dt^2}, \ldots, \frac{d^k\mathbf{y}}{dt^k}\right) = 0$$

To save on some writing, the derivative of $\mathbf{y}$ with respect to t, $\dfrac{d\mathbf{y}}{dt}$ is denoted by $\dot{\mathbf{y}}$. If $\mathbf{y} \in \mathbb{R}^d$, then we say we have a *system* of d ODEs. The ODE is said to be of *order* k if $\mathbf{F}$ depends on the kth derivative of $\mathbf{y}$ but not on any higher derivatives. If you can solve for the derivative of the highest order so as to write:

$$\frac{d^k\mathbf{y}}{dt^k} = \mathbf{G}\left(t, \mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \ldots, \frac{d^{k-1}\mathbf{y}}{dt^{k-1}}\right),$$

then the ODE is called *explicit*. Otherwise it is called *implicit*. In this case the coefficient of the highest derivative typically vanishes on some subset of the phase space, and the ODE is said to have "singularities". If we have an explicit ODE, we can rewrite it as a *system* of first order equations by defining new variables

$$\mathbf{x}_1 := \mathbf{y}, \quad \mathbf{x}_2 := \dot{\mathbf{y}}, \quad \ldots, \quad \mathbf{x}_i := \frac{d^{i-1}\mathbf{y}}{dt^{i-1}}, \quad \ldots, \quad \mathbf{x}_k := \frac{d^{k-1}\mathbf{y}}{dt^{k-1}}.$$

This results in a new system of first order equations

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ &\vdots \\ \dot{\mathbf{x}}_i &= \mathbf{x}_{i+1} \\ &\vdots \\ \dot{\mathbf{x}}_k &= \mathbf{G}\left(t, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\right). \end{aligned} \tag{1.1}$$

[1] I will typically try to single out definitions as needed in these notes, but in this first (presumably) review section as well as in the tutorial sheets, I will just be italicising them inline for the sake of brevity

(By the way, there are other ways of converting a system to first order and in many applications, these are *way* more convenient.)

Since each $\mathbf{x}_i$ represents d variables, we actually have a system of $n = kd$ variables. Thus, each kth order system of ODE's on $\mathbb{R}^d$ is a first order system on $\mathbb{R}^n$. Equation (1.1) is a special case of a general system of first order ODEs,

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n$$

which can be written even more compactly as

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) \tag{1.2}$$

For a bit, we'll use the $\mathbf{x}$ notation (boldface) to denote that we're considering $\mathbf{x}$ to be a vector, but after a while, we'll drop this and get around it by specifying clearly the domain and range of our functions, and letting the context make it clear what we're talking about. When f is independent of t equation (1.2) is called *autonomous* and we can simplify (1.2) even further to

$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{1.3}$$

For the system (1.3), the function f specifies the 'velocity' at each point in the phase space (or domain of f). This is called a *vector field.*

In principle we can reduce a non–autonomous system to an autonomous one by introducing a new variable $x_{n+1} = t$ and so we have $\dot{x}_{n+1} = 1$, and we have a new system of equations defined on space of one higher dimension. However, in practice, sometimes it is better to think about the autonomous and non–autonomous case separately

Often we are interested in solutions to ODE's that start at a specific initial state, so $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ for example. These are called *Cauchy problems* or *initial value problems.* What is awesome is that you can pretty much always numerically compute (over a short enough time anyway) the solution to a Cauchy problem. However, the bad news is that it is almost impossible to write down a closed form for an analytic solution to a Cauchy problem. We can however, (by being clever and hardworking) devise methods for analysing the solutions to ODEs without actually solving them. This is a reasonable approximation of a definition of what the field of *continuous-time dynamical systems* is about. (It is of course about much more than this, I just wanted to give succinct paraphrasing of how someone who works in the area of continuous time dynamical systems might describe their work).

Let's do some examples illustrating how to pass from an explicit ODE to a first order system.

Example 1.1. Consider the following (explicit) third order ODE:

$$\dddot{y} - (\ddot{y})^2 + \dot{y}y + \cos(t) = 0 \tag{1.4}$$

Here $y(t)$ is a function of the real variable t. We can rewrite eq. (1.4) as

$$\dddot{y} = (\ddot{y})^2 - \dot{y}y - \cos(t) =: G(t, y, \dot{y}, \ddot{y}) \tag{1.5}$$

Now we define new dependent variables (i.e. functions of t) x_1, x_2, and x_3 as

$$x_1(t) := y(t), \quad x_2(t) := \dot{y}(t) \quad x_3(t) := \ddot{y}(t)$$

Now we have a new *first order system* of equations

$$\begin{aligned} \dot{x}_1 &= \frac{d}{dt}y = \dot{y} = x_2 \\ \dot{x}_2 &= \frac{d}{dt}\dot{y} = \ddot{y} = x_3 \\ \dot{x}_3 &= \frac{d}{dt}\ddot{y} = \dddot{y} = x_3^2 - x_2x_1 - \cos(t) = G(t, x_1, x_2, x_3) \end{aligned} \tag{1.6}$$

Now if we let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ then we can succinctly write eq. (1.6) as

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

where $f(\mathbf{x})$ is the function $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(t, x_1, x_2, x_3) = (x_2, x_3, G(t, x_1, x_2, x_3))$$

Example 1.2. Consider the second order system of ODEs given by

$$\ddot{\mathbf{y}} + \mathbb{A}\mathbf{y} = 0 \tag{1.7}$$

where $\mathbf{y} = (y_1, y_2)^\top$ is a vector of functions in $\mathbb{R}^2$ and $\mathbb{A}$ is a 2×2 matrix with real entries, say

$$\mathbb{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To get a feel for the utility of the notation, writing this out, we have the following system of ODEs

$$\begin{aligned} \ddot{y}_1 + ay_1 + by_2 &= 0 \\ \ddot{y}_2 + cy_1 + dy_2 &= 0 \end{aligned} \tag{1.8}$$

To write this out as a *first order* system, we follow the prescription given in the first section. We set $\mathbf{x}_1 := \mathbf{y}$ and so if $\mathbf{x}_1 = (x_1, x_2)^\top := (y_1, y_2)^\top$, we have $x_1 = y_1$ and $x_2 = y_2$ and we set $\mathbf{x}_2 := \dot{\mathbf{y}}$ so if $\mathbf{x}_2 = (x_3, x_4)^\top$, then $x_3 := \dot{y}_1$ and $x_4 := \dot{y}_2$. Now we use eq. (1.7) and the defining relations for x_i to get a 4×4 first order system

$$\begin{aligned} \dot{x_1} &= x_3 \\ \dot{x_2} &= x_4 \\ \dot{x_3} &= -ax_1 - bx_2 \\ \dot{x_4} &= -cx_1 - dx_2. \end{aligned} \tag{1.9}$$

It is possible to write this more compactly (still as a first order system).

$$\begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{A} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{1.10}$$

where $\mathbb{I}$ is the 2×2 identity matrix. Or even more simply if we set $\mathbf{x} := (\mathbf{x}_1, \mathbf{x}_2)^\top = (x_1, x_2, x_3, x_4)$ ($= (y_1, y_2, \dot{y}_1\dot{y}_2)$ in our original dependent variables), then we can write $\dot{\mathbf{x}} = \mathbb{B}\mathbf{x}$, where $\mathbb{B}$ is the 4×4 matrix on the right hand side of eq. (1.10).

2. The First Examples

For the most part this class is about the following equation (in some form or another)

$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{2.1}$$

Here $\dot{\mathbf{x}}$ means $\dfrac{d\mathbf{x}}{dt}$, $\mathbf{x}(t)$ is (typically) a vector of functions $\mathbf{x} = (x_1(t), \dots, x_n(t))$ with $x_i(t) : \mathbb{R} \to \mathbb{R}$, and f is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. We'll begin our study of ordinary differential equations (and hence of (2.1)) with the simplest types of f.

Example 2.1 (The function $f(\mathbf{x}) = 0$)**.** Let's start by considering the simplest function possible for the right hand side of eq. (2.1), $f(\mathbf{x}) \equiv 0$. In this case, there are no dynamics at all, and so if $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$, then eq. (2.1) becomes

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0, \quad \cdots, \quad \dot{x}_n = 0. \tag{2.2}$$

The *solution* to our ODE in this case will be an n-tuple of functions $(x_1(t), \dots x_n(t))$ which simultaneously satisfy $\dot{\mathbf{x}}(t) = 0$. It is easy to solve the ODE in this case, we just integrate each of the equations in eq. (2.2) once to get

$$x_1(t) = c_1, \quad \cdots, \quad x_n(t) = c_n, \tag{2.3}$$

where the c_i's are the constants of integration. Or more succinctly, we have

$$\mathbf{x}(t) = \mathbf{c}$$

where $\mathbf{c} \in \mathbb{R}^n$ is a constant vector. For what it's worth, we remark here that all possible solutions to $\dot{\mathbf{x}} = 0$ are of this form, and that the set of all possible solutions, i.e. the *solution space* is a vector space $\approx \mathbb{R}^n$.

Example 2.2 (The function $f(\mathbf{x}) = \mathbf{c}$)**.** A slightly (though not much) more complicated example is when the right hand side of eq. (2.1) is a constant function, or constant vector in $\mathbf{c} \in \mathbb{R}^n$. In this case we have

$$\dot{x}_1 = c_1, \quad \dot{x}_2 = c_2, \quad \cdots, \quad \dot{x}_n = c_n. \tag{2.4}$$

Just as before, we can integrate these equations once more to get

$$x_1(t) = c_1 t + d_1, \quad \cdots, \quad x_n(t) = c_n t + d_n, \tag{2.5}$$

where the d_i's are the constants of integration this time. Again, we remark that the dimension of the set of solutions is n. The solutions don't form a vector space per se, as the sum of two solutions is not again a solution. However, the set of solutions does contain a vector space of dimension n.

Related to this (and essentially just as simple) is when the right hand side of eq. (2.1) is independent of $\mathbf{x}$. In this case, we can again integrate each vector component separately to solve our system. For example, suppose the right hand side were $F(t) = (f_1(t), f_2(t), \dots, f_n(t))$. Our system of equations would then be

$$\dot{x}_1 = f_1(t), \quad \dot{x}_2 = f_2(t), \quad \cdots, \quad \dot{x}_n = f_n(t), \tag{2.6}$$

and we could solve each equation independently by simply finding the anti-derivative (if possible).

3. Matrix ODEs

Now we're going to move on from the simplest examples, to the next simplest type of f. We want to study eq. (2.1) when f is a linear map in the dependent variables x_i. Such systems are called *linear ODEs* or *linear systems.* 'Recall' the following:

Definition 3.1. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *linear* if the following hold

(1) (superposition) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$
(2) (linear scaling) $f(c\mathbf{x}) = cf(\mathbf{x})$ for all $c \in \mathbb{R}$ and $\mathbf{x}$ in $\mathbb{R}^n$.

Remark. A quick aside, Example 2.1 is an example of when $f(\mathbf{x})$ is a linear map, while Example 2.2 is *not* an example of $f(\mathbf{x})$ being a linear map (in fact both properties in Definition 3.1 fail - try to see why). It is close though and sometimes it is called an *affine* map.

As you should already know, a linear map (sometimes called a linear transformation) $f : \mathbb{R}^n \to \mathbb{R}^n$, can be represented as a matrix once you choose a basis. If we let A denote the $n \times n$ matrix of the linear transformation f, this transforms eq. (2.1) into

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{3.1}$$

If A does not depend on t, we say it is a *constant coefficient* matrix. For the most part we'll consider A to be a real valued matrix, although this is not strictly necessary. In fact, almost everything about this course can be translated to work over $\mathbb{C}$, the complex numbers (some things quite easily, some not so much). For now, we will consider primarily constant coefficient matrices.

Example 3.1. Let's define the following matrices:

$$A_1 = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad A_3 = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of eq. (3.1) this means, say for A_1, that $\mathbf{x}$ is two dimensional (because A_1 is 2×2) and eq. (3.1) becomes

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -4x_1(t) - 2x_2(t) \\ x_1(t) - x_2(t) \end{pmatrix}. \tag{3.2}$$

So we have a system of 2 linear ordinary differential equations. Solving this system amounts to simultaneously finding functions $x_1(t)$ and $x_2(t)$ that satisfy eq. (3.2).

> **Question:** How do we go about solving the equation $\dot{\mathbf{x}} = A_i\mathbf{x}$?

The standard technique for solving linear ODEs involves finding the eigenvalues and eigenvectors of the matrix A. 'Recall' the following

Definition 3.2. An *eigenvalue* λ of an $n \times n$ matrix A is a complex number λ such that the following is satisfied

$$A\mathbf{v} = \lambda\mathbf{v}$$

where $\mathbf{v}$ is some non zero vector in $\mathbb{C}^n$ called an *eigenvector*.

This equation has a solution if and only if the matrix $A - \lambda\mathbb{I}$ (with $\mathbb{I}$ being the $n \times n$ identity matrix) is singular, that is if and only if

$$\rho(\lambda) := \det(A - \lambda\mathbb{I}) = 0.$$

Definition 3.3. The polynomial $\rho(\lambda)$ is an nth degree polynomial in λ and is called the *characteristic polynomial* of A.

Theorem 3.1 (The Fundamental Theorem of Algebra)**.** *The characteristic polynomial $\rho(\lambda)$ of an $n \times n$ matrix A has exactly n complex roots, counted according to their algebraic multiplicity.*

'Recall' the following definition:

Definition 3.4. The *algebraic multiplicity* of an eigenvalue λ is the largest integer k such that the characteristic polynomial can be written $\rho(r) = (r - \lambda)^k q(r)$, where $q(\lambda) \neq 0$. If λ is an eigenvalue of algebraic multiplicity equal to 1, it is called a *simple* eigenvalue.

We also have:

Definition 3.5. The number of linearly independent eigenvectors corresponding to the eigenvalue λ is called the *geometric multiplicity* of the eigenvalue.

You might remember the following

Proposition 3.1. *The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.*

Proof. DIY. □

Example 3.2. To find the eigenvalues of the matrix A_1 described above we take the determinant of $A_1 - \lambda I$ and find the roots of the characteristic equation

$$\rho(\lambda) = \begin{vmatrix} -4-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3)$$

Setting $\rho(\lambda)$ equal to zero and solving for λ we conclude that the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. To find the eigenvectors, we substitute in the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$ for λ and find the kernel (null space) of $A - \lambda\mathbb{I}$. Writing this out explicitly gives

$$\begin{pmatrix} -4+2 & -2 \\ 1 & -1+2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4+3 & -2 \\ 1 & -1+3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

which have kernels spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The reason that we are interested in finding the eigenvalues and the eigenvectors is that they enable us to find 'simple' (and eventually all) solutions to $\dot{\mathbf{x}} = A\mathbf{x}$. Suppose that $\mathbf{v}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ. Now consider the vector of functions $\mathbf{x}(t) = c(t)\mathbf{v}$, where $c(t)$ is some scalar valued function of t that we'll determine later. If we suppose that $\mathbf{x}(t)$ solves eq. (3.1), that is $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, then we must have

$$\dot{c}(t)\mathbf{v} = Ac(t)\mathbf{v} = c(t)A\mathbf{v} = c(t)\lambda\mathbf{v}$$

As $\mathbf{v}$ is nonzero, this means that $c(t)$ must satisfy $\dot{c}(t) = \lambda c(t)$. But we can solve this equation! We have that $c(t) = e^{\lambda t}$, and we get a solution to our original equation, eq. (3.1), namely $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.

Remark. A quick aside here. What we just did - guess at a simple form of a solution and plug it in and see where that leads us - is a fairly common technique in the study of differential equations. Such a guess-solution is called an *ansatz*, a word of German origin (Google tells me it means 'approach' or 'attempt'), and we will come back to using them (ansatzes) whenever they are useful.

Example 3.3. Returning to our example with A_1, we have, for each linearly independent eigenvector, an *eigensolution*:

$$\mathbf{x}_1(t) = e^{-2t}\mathbf{v}_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-3t}\mathbf{v}_2 = \begin{pmatrix} 2e^{-3t} \\ -e^{-3t} \end{pmatrix}.$$

Continuing on, we have that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent basis of eigenvectors of $\mathbb{R}^2$ (why?), and as our map f from eq. (2.1) earlier is linear (remember it's the matrix A_1), we have that if $\mathbf{x}_1(t)$ is a solution, and $\mathbf{x}_2(t)$ is a solution, then so is, by superposition and scaling, $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ for any constants c_1 and c_2. This means in particular, that our solution space (the set of all solutions) contains a vector space! Besides being pretty cool in its own right, this also enables us to write down solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ in the following (matrix) way:

$$\mathbf{x}(t) = \begin{pmatrix} e^{-2t} & 2e^{-3t} \\ -e^{-2t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{3.3}$$

The other nice thing about this formulation (that we'll prove in a week or so) is that this enables us to write *all* of the solutions to $\dot{\mathbf{x}} = A\mathbf{x}$. That is, our solution space not only contains this 2-dimensional vector space, that is all it contains.

It turns out that the process we just used in Example 3.3 generalises very nicely for almost all (whatever that means) $n \times n$ matrices.

4. Diagonalization and the Exponential of a Matrix

Suppose that A is an $n \times n$ matrix with complex entries. The goal of this subsection is to understand what is meant by the following:

$$e^A. \tag{4.1}$$

In terms of matrices, this is pretty straightforward and goes basically exactly how you would expect it to go. First the technical definition:

Definition 4.1. Suppose that A is an $n \times n$ matrix with real entries. Then we define (purely formally at this point) the *exponential* of A denoted $\exp(A)$ or e^A by

$$\exp(A) = e^A := \mathbb{I} + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \frac{1}{4!}A^4 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Why is this definition defined only purely formally? Well, for starters, we don't even know if it converges in each entry. IF it does, then it is straightforward to see that e^A is an $n \times n$ matrix itself. We'll start by describing a large class of matrices for which it is easy to see that e^A exists (that is, we have convergence in each of the entries), and in the process, learn (in theory anyway) how to compute it for relatively small matrices. In any case, we'll tackle some of the theory and see what, if anything, we can get out of it.

Remark. Computing e^A becomes pretty tricky (even if you know that it exists) as the size of A increases - actually even for relatively small matrices. There are quite a few reasons for this. There is a seminal work on the matter called *Nineteen Dubious Ways to Compute the Exponential of a Matrix*, by C. Moler and D. van Loan and a famous update on it twenty-five years later.

We're going to begin with a hypothesis that makes our task tractable. It is important to note that what follows will most emphatically *not* work for any old matrix! That is why I am putting the hypothesis in a separate box.

> **Hypothesis:**
> Suppose that our $n \times n$ matrix A had a set of n linearly independent eigenvectors. (The *same* n as the size of the matrix).

If we denote the eigenvectors of our matrix A by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, we can form a matrix whose columns are the eigenvectors of A. Denoting this matrix by P we have:

$$P := \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}.$$

Now for each $\mathbf{v}_i$ we have that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for the appropriate eigenvalue λ_i. Putting this together with our definition of P and the rules of matrix multiplication we have

$$\begin{aligned} AP &= \begin{pmatrix} | & | & & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 & \dots & A\mathbf{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \dots & \lambda_n\mathbf{v}_n \\ | & | & & | \end{pmatrix} \\ &= P \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = P\mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n) =: P\Lambda, \end{aligned}$$

where we have denoted the diagonal matrix with entries λ_i as $\mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$ and also as Λ, mostly because it will be convenient for writing later on.

To reiterate, **provided the matrix A has enough linearly independent eigenvectors**, then we can change basis and write

$$AP = P\Lambda \tag{4.2}$$

where Λ is a diagonal matrix with entries equal to the eigenvalues of the matrix A.

Now, since the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent, we have that the matrix P is invertible. This means that we can rewrite eq. (4.2) as

$$P^{-1}AP = \Lambda. \tag{4.3}$$

Definition 4.2. A matrix A that can be written in the form (equivalent to eq. (4.3))

$$A = P\Lambda P^{-1} \tag{4.4}$$

where P is an invertible matrix and Λ is a diagonal matrix is called (for obvious reasons) *diagonalizable* or (for less obvious reasons) *semisimple*.

Let's see what happens if we (formally) take the exponent of each side of eq. (4.3). We'll begin with the right hand side of eq. (4.3).

$$\begin{aligned}
\exp(\Lambda) = e^{\Lambda} &= \mathbb{I} + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{6}\Lambda^3 + \cdots + \frac{1}{k!}\Lambda^k + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k \\
&= \begin{pmatrix} 1 + \lambda_1 + \cdots + \frac{1}{k!}\lambda_1^k + \cdots & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 + \lambda_n + \cdots + \frac{1}{k!}\lambda_n^k + \cdots \end{pmatrix}
\end{aligned}$$

which you might recognize as

$$\begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} = \operatorname{diag}\left(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}\right)$$

So what we've just shown is that the exponent of a diagonal matrix

(1) always exists, and
(2) consists of e to the entries of the matrix.

What about the left hand side of equation (4.3)? Well, let's write it out and see what we get:

$$\begin{aligned}
e^{P^{-1}AP} &= \mathbb{I} + P^{-1}AP + \frac{1}{2}(P^{-1}AP)^2 + \frac{1}{3!}(P^{-1}AP)^3 + \cdots \\
&= \mathbb{I} + P^{-1}AP + \frac{1}{2}P^{-1}APP^{-1}AP + \frac{1}{3!}P^{-1}APP^{-1}APP^{-1}AP + \cdots \\
&= P^{-1}\left(\mathbb{I} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots\right)P \\
&= P^{-1}e^{A}P.
\end{aligned}$$

Now equating these two sides and rearranging gives us what we were after in the first place

$$e^{A} = Pe^{\Lambda}P^{-1} \tag{4.5}$$

Example 4.1. Let's compute an example using the matrix A_1 from above. We have that

$$A_1 = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$
$$\text{so} \quad P^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Now e^{Λ} is easy to compute. It is just $\operatorname{diag}(e^{-2}, e^{-3})$. So we can compute e^{A_1} exactly by using formula (4.5)

$$e^{A_1} = Pe^{\Lambda}P^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{-3} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

which, when the dust settles gives

$$e^{A_1} = \begin{pmatrix} 2e^{-3} - e^{-2} & 2e^{-3} - 2e^{-2} \\ -e^{-3} + e^{-2} & -e^{-3} + 2e^{-2} \end{pmatrix}.$$

Returning to the theory, what eq. (4.5) has just shown is

Theorem 4.1.

(1) If the $n \times n$ matrix A is diagonalizable, (i.e. there are enough linearly independent eigenvectors), definition 4.1 is well-defined. That is, every element in the matrix e^A converges, provided A has enough linearly independent eigenvectors.

(2) If the $n \times n$ matrix A is diagonalizable, the eigenvectors of the matrix e^A are the same *as those of A and further the eigenvalues of e^A are just e^{λ_i}, the eignevalues of A raised to the power e.*

Question. Wait, what? Did we really just show Theorem 4.1 part (2)? Yes we did - prove this.

Okay so, we have a condition in our theorem, the 'provided the matrix A is diagonalizable' part.

Definition 4.3. An $n \times n$ matrix A is called *defective* if it has less than n linearly independent eigenvectors. That is, if it has an eigenvalue whose algebraic multiplicity is strictly greater than its geometric multiplicity.

Questions. How severe of a restriction is this? Are there a lot of matrices that have 'enough' linearly independent eigenvectors? Does e^A exist if A is defective? If it does, is it possible to compute e^A?

It turns out that being semisimple isn't that bad of a restriction, and moreover, it doesn't matter anyway. So the answer to all three questions is positive. 'A lot' of matrices are diagonalizable, but it doesn't matter since you can compute e^A for all $n \times n$ matrices A.

As a way to answering the first question we consider the following proposition.

Proposition 4.1. *If λ_1 and λ_2 are distinct eigenvalues of a matrix A with corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively, then $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent.*

Proof. Suppose for distinct eigenvalues $\lambda_1 \neq \lambda_2$, we had a dependence relation on the corresponding eigenvectors. So we had some nonzero constants k_1 and k_2 such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = 0 \tag{4.6}$$

Suppose, without loss of generality that $k_2 \neq 0$. Applying A to both sides of eq. (4.6) gives

$$Ak_1\mathbf{v}_1 + Ak_2\mathbf{v}_2 = k_1\lambda_1\mathbf{v}_1 + k_2\lambda_2\mathbf{v}_2 = 0 \tag{4.7}$$

and multiplying both sides of eq. (4.6) by λ_1 gives

$$k_1\lambda_1\mathbf{v}_1 + k_2\lambda_1\mathbf{v}_2 = 0. \tag{4.8}$$

Subtracting eq. (4.8) from eq. (4.7) gives

$$(\lambda_2 - \lambda_1)k_2\mathbf{v}_2 = 0. \tag{4.9}$$

but this means that $\lambda_2 = \lambda_1$, a contradiction, proving the proposition. □

Why does this proposition give some insight into the answer to the question of how many defective matrices are there? Well, it tells us that if we have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then we'll have n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. But the eigenvalues are the roots of the characteristic polynomial - indeed we can always write the characteristic polynomial as

$$\rho(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n),$$

with perhaps some λ_is repeated according to their multiplicity. If we assume there aren't any eigenvalues with multiplicity greater than 1, then a partial answer to our question is found in the answer to the following: "how many polynomials of degree n have n distinct (and hence simple) roots?" The answer is "most of them" although this is sort of difficult at this point to make precise, but we can explore it a little through the example of 2×2 matrices.

Example 4.2. The space of 2×2 matrices with real coefficients is four dimensional and a general matrix can be written

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, b, c, d being unknown. The characteristic polynomial for a general 2×2 matrix is given by

$$\rho(\lambda) = \lambda^2 - (a + d)\lambda + ad - bc$$

and so you notice that coefficients of the polynomial are the trace of A which we will denote as τ and the determinant, which we'll denote as δ. Now the question of how many defective 2×2 matrices are there is reduced to that of how many polynomials of the form $x^2 - \tau x + \delta$ have multiple roots (actually this will just be an upper bound, as we know that a characteristic polynomial of a matrix can have multiple roots, but the matrix might not be defective). The roots of $\rho(\lambda)$ are given by the quadratic formula (sing the song):

$$\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

and you can see that the only way we can have multiple roots of the polynomial $\rho(x)$is if $\tau^2 - 4\delta = 0$. So how often does this occur? Well, there are a couple of ways to think about it.

The first way is to consider this equation in (τ, δ) space. We see right away that this is the equation of a parabola - so we have a (one dimensional) curve in (τ, δ) space (which is 2-dimensional) - that isn't very many. In terms of matrices, this means that so long as the trace and determinant stay away from this curve, then we're fine - our matrix isn't defective. Since 'most' values of τ and δ don't lie on this curve, we can infer that 'most' matrices aren't defective, and therefore 'most' 2×2 matrices are diagonalizable. See Figure 1 below.

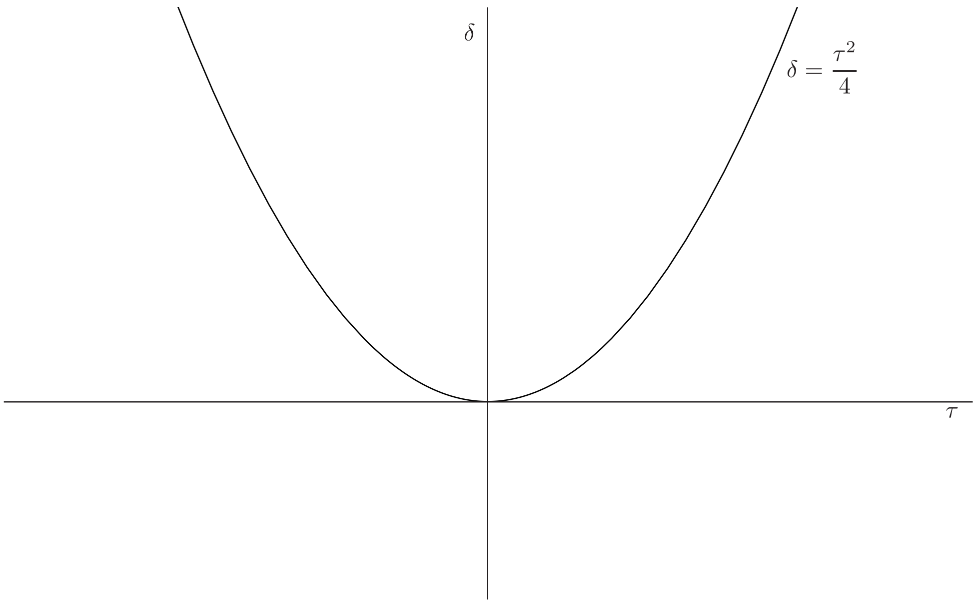


FIGURE 1. A plot of the curve $\tau^2 = 4\delta$ in the (τ, δ) plane. 2×2 matrices with trace and determinant values not on the parabola have distinct eigenvalues, and hence their eigenvectors form a linearly independent set of $\mathbb{R}^2$, and hence they are diagonalizable.

The second way is to go back to the coefficients of the matrices themselves, and look at the equation relating τ and δ in terms of these. This gives

$$0 = \tau^2 - 4\delta = (a + d)^2 - 4(ad - bc) = a^2 + 2ad + d^2 - 4ad + 4bc = (a - d)^2 + 4bc,$$

which is a single equation in the variables a, b, c, d. What this equation does, is it carves out a 3-d 'hypersurface' in the 4-d space of 2×2 matrices. Since 'most' of the 2×2 matrices will avoid this surface, we conclude that 'most' of the 2×2 matrices are diagonalizable.

Question. By the way, this is only part of the story. It is certainly possible for a matrix to have multiple eigenvalues, but linearly independent eigenvectors. The question now, is on this $\tau^2 - 4\delta = 0$ curve or the $(a - d)^2 + 4bc$ hypersurface, how many (is it 'most' or not) of the matrices are defective? (We will answer this later).

Now, let's move on to showing that needing enough eigenvectors doesn't matter. That is, every matrix A has an exponential e^A. In order to do this, we need a few things. First recall the following

Definition 4.4. Suppose A and B are two $n \times n$ matrices. We say that A and B *commute* if $AB = BA$ or, equivalently, $AB - BA = 0$.

Next is the proposition,

Proposition 4.2. *If A and B are $n \times n$ commuting matrices, then*

$$e^{(A+B)} = e^A e^B. \tag{4.10}$$

Proof. In principle, we should add the caveat 'provided both sides of the equation exist' although this is not really necessary for a couple of reasons. The first is that we shall see soon enough that an exponential exists for all $n \times n$ matrices. Secondly, since this proof is purely formal algebraic manipulation, we can (sort of) say the statement of the proposition, even if the matrices didn't converge (though we'd definitely need to be careful to say that this was only a *formal* equivalence). In any case, the proof is by direct computation and manipulation:

$$\begin{aligned} e^{(A+B)} &= \mathbb{I} + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots \\ &= \mathbb{I} + (A+B) + \frac{1}{2}A^2 + \frac{1}{2}AB + \frac{1}{2}BA + \frac{1}{2}B^2 + \frac{1}{3!}A^3 + \cdots \\ &= \mathbb{I} + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 + \frac{1}{3!}A^3 + \frac{1}{2}A^2B + \frac{1}{2}AB^2 + \cdots \qquad (*) \end{aligned}$$

because $AB = BA$.

Now we write out the right hand side of eq. (4.10) and repeatedly use the fact that $AB = BA$:

$$e^A = \mathbb{I} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots \quad \text{and} \quad e^B = \mathbb{I} + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots$$

$$\begin{aligned} e^A e^B &= \left(\mathbb{I} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots\right)\left(\mathbb{I} + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots\right) \\ &= \mathbb{I} + (A+B) + \left(\frac{1}{2}A^2 + AB + \frac{1}{2}B^2\right) + \left(\frac{1}{3!}A^3 + \frac{1}{2}A^2B + \frac{1}{2}AB^2 + \cdots\right) + \cdots \end{aligned}$$

which is the same as $(*)$. □

It is worth noticing that you really do need the fact that $AB = BA$. Almost any pair of noncommuting matrices will not satisfy the equation $e^{A+B} = e^A e^B$.

It is also interesting (maybe) to have a look at the converse. That is, suppose that $e^A e^B = e^{(A+B)}$ for some $n \times n$ matrices A and B, does that mean that $AB - BA = 0$ (or that $[A, B] = 0$ to use the *commutator* notation)? The short answer is 'no', although the full answer is a bit less clear. We have the following counterexample to the converse of Proposition 4.2.

Example 4.3. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi i \end{pmatrix}$ where $i = \sqrt{-1}$ is the complex number with positive imaginary part whose square is -1. Then let $B = \begin{pmatrix} 0 & 1 \\ 0 & 2\pi i \end{pmatrix}$. It is pretty straightforward to see that $e^A = e^B = e^{(A+B)} = \mathbb{I}$, however we have that $AB = \begin{pmatrix} 0 & 0 \\ 0 & -4\pi^2 \end{pmatrix}$, while $BA = \begin{pmatrix} 0 & 2\pi i \\ 0 & -4\pi^2 \end{pmatrix}$.

Example 4.4. It is easy enough to construct *real* examples if we want. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\pi \\ 0 & 0 & 2\pi & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2\pi \\ 0 & 0 & 2\pi & 0 \end{pmatrix}$$

Then you can verify for yourself that $e^A e^B = e^{(A+B)}$ but that $AB \neq BA$.

Example 4.5. Here is another, slightly more complicated example. Suppose that we define A and B as the following

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \neq b \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We can directly compute

$$e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}, \quad e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \quad e^{A+B} = \begin{pmatrix} e^a & \frac{e^a - e^b}{a-b} \\ 0 & e^b \end{pmatrix}.$$

So we will have that $e^A e^B = e^{A+B}$ when we can find a, b such that

$$(a-b)e^a = e^a - e^b.$$

If we set $x = (a-b)$, this means that we must solve $x = 1 - e^{-x}$. There are plenty of (complex) nonzero solutions to this which you can find using your favourite computer software package. (I used Mathematica and got $x \approx -6.60222 - 736.693i$.)

To summarise what we have said so far:

(1) $[A, B] = 0 \Rightarrow e^A e^B = e^{A+B}$ Always (Proposition 4.2).
(2) $e^A e^B = e^{A+B} \Rightarrow AB - BA = 0$ No.

The question then becomes: what other property can you put on the matrices A and B to change the second point to a 'yes'? One such property is the following

Proposition 4.3. *Suppose that A and B are real, symmetric (that is $A^T = A$) $n \times n$ matrices, then $e^A e^B = e^{A+B} \Rightarrow [A, B] = 0$.*

The only proofs that I can find to this proposition are beyond the scope of this course, but it might be an interesting problem to try and prove this yourself, using what you know.

The condition that A and B be real, symmetric matrices is quite a strong one, and we would like to know if there is a weaker condition that we might apply that would also make the second statement (or the converse of proposition 4.2) true. The answer to this is (I believe) an open problem. Indeed, it is not even clear to me whether or not the converse is true for real 2×2 matrices. That is, consider the following: Suppose A and B are real 2×2 matrices, then does $e^A e^B = e^{A+B} \Rightarrow AB - BA = 0$? In all of the 2×2 counterexamples to Proposition 4.2 we had, our matrices had complex entries, so it is conceivable that requiring the matrices to be real would mean that $e^A e^B = e^{A+B} \Rightarrow AB - BA = 0$ (though I know of no proof of this).

Continuing on:

Definition 4.5. A matrix is called *nilpotent* if some power of it is 0. The smallest (integer) power of a nilpotent matrix is called its *degree of nilpotency* (or sometimes just its *nilpotency*.

Example 4.6. For example in the following matrix

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N^3 = 0.$$

This means that the (formal) exponential series for the matrix N stops after a finite (here 3) number of steps. We have $e^N = \mathbb{I} + N + \frac{1}{2}N^2$, or

$$e^N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now are going to appeal to a theorem from linear algebra that will guarantee us the ability to take the exponential of any matrix.

Theorem 4.2 (Jordan Canonical Form)**.** *If A is an $n \times n$ matrix with complex entries then there exists a basis of $\mathbb{C}^n$ and an invertible matrix B consisting of those basis vectors such that the following holds:*

$$B^{-1}AB = S + N$$

where the matrix S is diagonal and the matrix N is nilpotent and upper (or lower) triangular. Further $SN - NS = 0$, that is S and N commute.

We're not going to prove this, and for the moment, it doesn't tell us anything about *how* to compute the basis B. Later we're going to develop another algorithm in order to compute the exponential of a matrix, but in performing said algorithm we may (and often will) bypass the Jordan Canonical form. I just wanted to include this theorem so that one, you could see it, and two so that we see more directly why you can take the exponential of any matrix.

Given the statement of the theorem, we can now write down the exponential of any matrix A,

$$e^A = Be^S e^N B^{-1},$$

and since the right hand side converges (we've shown this), we must have that it is equal to the left hand side. (We effectively use this theorem to define the exponential of our defective matrices.) It is actually possible to show that the exponential of a matrix A exists for any A without this theorem, however, I wanted you to know this theorem, (some of you may come across it in a higher linear algebra class but it is really a fundamental theorem of linear algebra, so everyone should be exposed to it at least once).

Example 4.7. Now that we have the exponential of any matrix, we can consider the matrix e^{At}, where t is some real number. This is just defined by the series that you would expect, and keeping track of the fact that A is a matrix and t is a scalar.

$$e^{At} = \mathbb{I} + At + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

When $t = 1$ this is just the original exponential series for e^A. We have that e^{At} makes sense for all t and all $n \times n$ matrices A, and so for any fixed $n \times n$ matrix A, we have a map $e^{At} : \mathbb{R} \to M_n$ from the reals to the space of $n \times n$ matrices. Further, we have for any other $s \in \mathbb{R}$ we can write $AtAs = AsAt$, so we have that

$$e^{At}e^{As} = e^{At+As} = e^{A(t+s)}.$$

In fact, we have that e^{At} is *invertible* for all matrices A and all real t - can you show this?

5. Complex Eigenvalues

To begin the section on complex eigenvalues, let us consider a couple of useful examples.

Example 5.1. Let J be the 2×2 matrix given by $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is not hard to show that $J^2 = -\mathbb{I}$, $J^3 = -J$ and $J^4 = \mathbb{I}$. Using these facts we can directly compute that

$$\begin{aligned} e^{Jt} &= \mathbb{I}\sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!} + J\sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{(2m+1)!} \\ &= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \end{aligned}$$

which some of you might recognize as a rotation matrix of the plane by t degrees. The matrix J is the matrix representation of the complex number $i = \sqrt{-1}$ and the above formula is the 2×2 matrix analog of the theorem $e^{it} = \cos t + i \sin t$. We explore this idea further in the following example.

Example 5.2. The matrix representation of the number i by the matrix J can be extended to consider the entire complex plane. That is, for each complex number $z = x + iy$ we define a 2×2 matrix M_z, and we will have that multiplication and addition are preserved. For each $z \in \mathbb{C}$ define the matrix M_z by $M_z := \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x\mathbb{I} + yJ$. Then it is straightforward to check that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we have that $M_{z_1} + M_{z_2} = M_{z_1+z_2}$. Further (and you should do this yourself) it is easy to see that $M_{z_1}M_{z_2} = M_{z_1z_2} = M_{z_2z_1} = M_{z_2}M_{z_1}$. We also claim that $M_{e^z} = e^{M_z} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$ for all $z \in \mathbb{C}$. This is easily established by the previous assertions and fact that $x\mathbb{I}$ and yJ commute. We have

$$M_{e^z} = M_{e^x}M_{e^{iy}} = e^{x\mathbb{I}}e^{yJ} = e^{x\mathbb{I}+yJ} = e^{M_z}.$$

This is pretty remarkable, but it also means that we can take the matrix exponent of matrices like M_z fairly easily. An intuitive reason for this is that we just have multiplication and addition in the original definition of matrix exponents, and commutativity of the matrices $\mathbb{I}$ and J. Thus we have that $e^z = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$.

A few more things to note about this representation of complex numbers:

(1) $M_{\bar{z}} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = M_z^T$, i.e. complex conjugation behaves as you would expect.
(2) If r is a real number $M_r = r\mathbb{I}$. This was used in the derivation of M_{e^z} but I wanted to make it explicit here.
(3) If ir is a purely imaginary number then $M_{ir} = rJ$.
(4) $M_zM_{\bar{z}} = M_{|z|^2} = |z|^2\mathbb{I}$.
(5) We have that $\det(M_z) = |z|^2$. The determinant give us a *norm* that we can use on our complex numbers.
(6) M_z is invertible and the inverse is given by $M_{\frac{1}{z}} = M_{\frac{\bar{z}}{|z|^2}} = \frac{1}{\det(M_z)}M_{\bar{z}} = \frac{1}{\det(M_z)}M_z^T$.

Some of these might come in handy later on.

In Example 5.1 we looked at $Jt = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$ and computed e^{Jt} by using the map to the complex numbers and by using the power series of the exponential. Now we're going to compute it using the diagonalization procedure that was outlined in Section 4. The eigenvalues of Jt are $\lambda = \pm it$ where $i = \sqrt{-1}$ and the eigenvectors are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$ respectively. So our matrix of eigenvectors is $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Further it is pretty straightforward to see that $P^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$. Lastly we set $\Lambda t = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}$, the matrix of eigenvalues. So we have that

$$e^{Jt} = Pe^{\Lambda t}P^{-1} = \frac{1}{2i}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \begin{pmatrix} \frac{e^{it}+e^{-it}}{2} & -\frac{e^{it}-e^{-it}}{2i} \\ \frac{e^{it}-e^{-it}}{2i} & \frac{e^{it}+e^{-it}}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Which is exactly what we got before (this is good). Now we remark that J is real, and for real t, so is Jt, so if we just plug it into the series for the exponential, then we are simply manipulating real matrices, with real steps. But the way we just did it requires a complex intermediate step. Evidently this is not maximally desirable. What we will be working towards now is a *real* normal form for a matrix of a linear transformation.

Suppose that A is a real $n \times n$ matrix. Then the characteristic polynomial of A, which we denote here by $\rho(\lambda) = \det(A - \lambda\mathbb{I})$ will only have real coefficients, so in particular if $\lambda = a + ib$ with $a, b \in \mathbb{R}$ is an eigenvalue of A, then so is $\bar{\lambda} = a - ib$. The other thing to notice here is that *eigenvectors* will also come in complex pairs, and moreover by taking the complex conjugate, the eigenvector corresponding to $\bar{\lambda}$ will be the complex conjugate of the eigenvector of λ. That is if $A\mathbf{v} = \lambda\mathbf{v}$, then $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$. Now suppose that $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ with $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$ is your complex eigenvector. Then we have that $2\mathbf{u} = \mathbf{v} + \bar{\mathbf{v}}$, and so we can show that

$$A2\mathbf{u} = A\mathbf{v} + A\bar{\mathbf{v}} = \lambda\mathbf{v} + \bar{\lambda}\bar{\mathbf{v}} = 2(a\mathbf{u} - b\mathbf{w})$$

So in particular $A\mathbf{u} = (a\mathbf{u} - b\mathbf{w})$. Likewise, you can show that $A\mathbf{w} = b\mathbf{u} + a\mathbf{w}$. Now we set P to be the $n \times 2$ matrix $P = \begin{pmatrix} | & | \\ \mathbf{u} & \mathbf{w} \\ | & | \end{pmatrix}$, and we have that $AP = P\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. This follows exactly what we did in the diagonalization procedure, except it allows us to find a 'normal form' for a real matrix with complex eigenvalues without a complex intermediate step. It should also be noted here that we have only done this really for 2 eigenvectors, but really find the exponential of a matrix, we need to do this for a 'full set'. We'll do an example first, and then some generalisations.

Example 5.3. Let's define the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

We will compute e^A and e^{At}. We have that the eigenvalues of A are $1 \pm i\sqrt{2}$ and 1 with corresponding eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} -1 \\ \mp i\sqrt{2} \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. This means that $\mathbf{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 0 \end{pmatrix}$, and so our change of basis matrix is $P = [\mathbf{u}, \mathbf{w}, \mathbf{v}_3]$. We have $P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -\sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$, and you can also check that $P^{-1} = \frac{1}{2}P$. Next, we consider the matrix $\Lambda := P^{-1}AP$. We can see straight away that

$$P^{-1}AP = \Lambda = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let's pause here for a notational bit. The matrix Λ is in what is called *block diagonal form*. Another (short-hand) way of writing this is the following. Let $\Lambda_1 := \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$, and $\Lambda_2 := 1$ (the 1×1 identity matrix), then $\Lambda = \Lambda_1 \oplus \Lambda_2$.

Remark. A note of caution - this symbol is often used in a different context in linear algebra as well - to denote the *direct sum* of two vector spaces. That is, if E_1 is the vector subspace of $\mathbb{R}^3$ spanned by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and E_2 is the vector subspace of $\mathbb{R}^3$ spanned by the vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$, then $E_1 \oplus E_2$ in this context would be the $2D$ subspace that consists of the (x, z) plane. The context should always make it clear whether we are talking about the direct sum of vector spaces or of matrices, but this remark is just to warn you to be careful until you're more familiar with what's going on.

Returning to our example, we have (in terms of matrices) $\Lambda = \Lambda_1 \oplus \Lambda_2$. The reason it is convenient to write it like this is the following proposition.

Proposition 5.1. *Suppose an $n \times n$ matrix A can be written in block diagonal form $A = B \oplus C$, then $e^A = e^B \oplus e^C$.*

Proof. We note that $A = B \oplus C$ means that $A = \begin{pmatrix}B & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & C\end{pmatrix}$ and that these last matrices commute. This means that e^A can be written as the products of the exponents of these two matrices. Writing this out we have

$$e^A = \begin{pmatrix}e^B & 0\\0 & \mathbb{I}\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\0 & e^C\end{pmatrix} = e^B \oplus e^C$$

□

Okay, so we can now apply Proposition 5.1 to Λ and get that $e^\Lambda = e^{\Lambda_1} \oplus e^{\Lambda_2}$, and likewise $e^{\Lambda t} = e^{\Lambda_1 t} \oplus e^{\Lambda_2 t}$. Now $e^{\Lambda_2 t} = e^t$. Pretty easy. To find out what $e^{\Lambda_1 t}$ is, we can either compute it directly (perhaps best if we're working in a vacuum) or we can notice that $\Lambda_1 t = \mathbb{I}t - \sqrt{2}Jt$ and that these matrices commute. So we can apply our results from the previous example to get that $e^{\Lambda_1} = e^t \begin{pmatrix}\cos\sqrt{2}t & \sin\sqrt{2}t\\-\sin\sqrt{2}t & \cos\sqrt{2}t\end{pmatrix}$. So, putting this all together we have that

$$e^{\Lambda t} = e^t \begin{pmatrix}\cos\sqrt{2}t & \sin\sqrt{2}t & 0\\-\sin\sqrt{2}t & \cos\sqrt{2}t & 0\\0 & 0 & 1\end{pmatrix},$$

which is real as we expected, and we got without a complex valued intermediate step. If we want to know what e^{At} is then we need to change back to our original basis. That is we have $e^{At} = Pe^{\Lambda t}P^{-1}$. This I will let you do yourselves as it is a bit messy should probably be done in Matlab or Mathematica.

We can now generalise this procedure. Suppose that you have n eigenvalues, of which the first $2k$ are complex. Write these as $\lambda_1 = a_1 + ib_1, \bar{\lambda}_1 = a_1 - ib_1, \ldots \lambda_k = a_k + ib_k, \bar{\lambda}_k = a_k - ib_k$, and the last $n - 2k$ of which are real $\lambda_{2k+1}, \ldots \lambda_n$. Suppose too that we have the associated eigenvectors $\mathbf{v}_1, \bar{\mathbf{v}}_1, \ldots, \mathbf{v}_k\bar{\mathbf{v}}_k, \mathbf{v}_{2k+1} \ldots \mathbf{v}_n$, and for the complex eigenvectors we have $\mathbf{v}_j = \mathbf{u}_j + i\mathbf{w}_j$. Then we can define the matrix P as follows

$$P = \begin{pmatrix}| & | & & | & | & | & & |\\ \mathbf{u}_1 & \mathbf{w}_1 & \ldots & \mathbf{u}_k & \mathbf{w}_k & \mathbf{v}_{2k+1} & \ldots & \mathbf{v}_n\\ | & | & & | & | & | & & |\end{pmatrix}.$$

Then (you might want to check this out for yourself) we have that

$$\Lambda := P^{-1}AP = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \oplus \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \oplus \cdots \oplus \operatorname{diag}(\lambda_{2k+1}, \ldots, \lambda_n).$$

This, with some straightforward manipulation of matrices, gives us a formula for e^{At}. We have that

$$e^{At} = Pe^{\Lambda t}P^{-1}.$$

Example 5.4. Let's do another example. Consider the matrix

$$A = \begin{pmatrix} 7 & 1 & 4 & -8 \\ 3 & 3 & 6 & -6 \\ 2 & -4 & 8 & -4 \\ 6 & 0 & 0 & 0 \end{pmatrix}.$$

Let's compute e^A and e^{At}. The eigenvalues of A are $\lambda_1, \bar{\lambda}_1 = 6 \pm 6i$ and $\lambda_2, \bar{\lambda}_2 = 3 \pm 3i$ and with eigenvectors $\mathbf{v}_1 \bar{\mathbf{v}}_1 = (1 \pm i, 1 \pm i, \pm i, 1)$ and $\mathbf{v}_2, \bar{\mathbf{v}}_2 = (1 \pm i, 1 \mp i, 2, 2)$ respectively. Thus we have that $a_1 = b_1 = 6$ and $a_2 = b_2 = 3$. We also have that $\mathbf{u}_1 = (1, 1, 0, 1)$ and $\mathbf{w}_1 = (1, 1, 1, 0)$ while $\mathbf{u}_2 = (1, 1, 2, 2)$ and $\mathbf{w}_2 = (1, -1, 0, 0)$. We set

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{w}_1 & \mathbf{u}_2 & \mathbf{w}_2 \\ | & | & | & | \end{pmatrix},$$

and so

$$P^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 2 & -4 & 2 \\ 2 & 2 & 2 & -4 \\ -1 & -1 & 2 & 2 \\ 3 & -3 & 0 & 0 \end{pmatrix}.$$

Thus we have that

$$\Lambda = P^{-1}AP = \begin{pmatrix} 6 & 6 & 0 & 0 \\ -6 & 6 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -3 & 3 \end{pmatrix} = M_{\bar{\lambda}_1} \oplus M_{\bar{\lambda}_2}.$$

So finally, putting this all together, we have $e^A = Pe^{\Lambda}P^{-1}$ where

$$\begin{aligned} e^{\Lambda} &= e^{M_{\bar{\lambda}_1} \oplus M_{\bar{\lambda}_2}} = e^{M_{\bar{\lambda}_1}} \oplus e^{M_{\bar{\lambda}_2}} = M_{e^{\bar{\lambda}_1}} \oplus M_{e^{\bar{\lambda}_2}} \\ &= e^6 \begin{pmatrix} \cos(6) & \sin(6) \\ -\sin(6) & \cos(6) \end{pmatrix} \oplus e^3 \begin{pmatrix} \cos(3) & \sin(3) \\ -\sin(3) & \cos(3) \end{pmatrix}. \end{aligned}$$

To find e^{At} use the fact that the eigenvalues of At are the eigenvalues of A multiplied by t and the eigenvectors are the same. Thus $\Lambda t = P^{-1}AtP$ and $e^{At} = Pe^{\Lambda t}P^{-1}$ where

$$e^{\Lambda t} = e^{6t} \begin{pmatrix} \cos 6t & \sin 6t \\ -\sin 6t & \cos 6t \end{pmatrix} \oplus e^{3t} \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}.$$

6. Multiple/Repeated Eigenvalues

In this section, we are going to tackle the last remaining challenge; what to do when A has repeated eigenvalues, and not enough linearly independent eigenvectors. To do this, we need the following definition.

Definition 6.1. Suppose that λ_j is an eigenvalue of a matrix A with algebraic multiplicity n_j. Then we define *the generalised eigenspace* of λ_j as

$$E_j := \ker\left[(A - \lambda_j \mathbb{I})^{n_j}\right].$$

There are basically two reasons why this definition is important. The first is that these generalised eigenspaces are *invariant* under multiplication by A. That is if $\mathbf{v} \in E_j$ for some generalised eigenspace, of our matrix A then $A\mathbf{v}$ is as well.

Proposition 6.1. *Suppose that E_j is a generalised eigenspace of the matrix A, then if $\boldsymbol{v} \in E_j$ so is $A\boldsymbol{v}$.*

Proof. Suppose $\mathbf{v} \in E_j$ for some generalised eigenspace for some eigenvector λ. Then $\mathbf{v} \in \ker(A - \lambda\mathbb{I})^k$ say for some k. Then we have that $(A - \lambda\mathbb{I})^k A\mathbf{v} = (A - \lambda\mathbb{I})^k (A - \lambda\mathbb{I} + \lambda\mathbb{I})\mathbf{v} = (A - \lambda\mathbb{I})^{k+1}\mathbf{v} + (A - \lambda\mathbb{I})^k \lambda\mathbf{v} = 0$. □

The second reason that we care about this is that it will turn out that this will give us a full set of linearly independent (generalised) eigenvectors, which we can use to decompose our matrix A. I am going to write this in a very general sense, but if you like, below, where it says T for linear transformation, you should think 'matrix' and where it says V for vector space, you can think $\mathbb{C}^n$ (or $\mathbb{R}^n$, but you have to be a little careful about what you mean by 'eigenspace' in this instance).

Theorem 6.1 (Primary Decomposition)**.** *Let $T : V \to V$ be a linear transformation of an n dimensional vector space over the complex numbers. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues (k is not necessarily equal to n). Let E_j be the generalised eigenspace corresponding to the eigenvalue λ_j. Then* $\dim(E_j) =$ *the algebraic multiplicity of λ_j and the generalised eigenvectors span V. That is, in terms of vector spaces we have*

$$V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

The proof of the previous theorem can be found in most texts on linear algebra. If you are interested, let me know, and I can track down a reference for you. This is basically what leads to the Jordan canonical form decomposition defined earlier.

Putting this together with the previous proposition, what this says is that the linear transformation T will decompose the vector space on which it acts into the direct sum of invariant subspaces. This is a *really* key idea, and we will revisit it often.

The next thing to do is to put all of this together to explicitly determine the semisimple nilpotent decomposition. Before, we had this matrix, Λ which was

(1) Block diagonal
(2) In a 'normal' form for complex eigenvalues
(3) Diagonal for real eigenvalues

So now, suppose we have a real matrix A with n (possibly complex) eigenvalues $\lambda_1, \dots, \lambda_n$ now repeated according to their multiplicity. Further suppose we break them up into the complex ones and the real ones, so we have $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_k, \bar{\lambda}_k$

complex eigenvalues and $\lambda_{2k+1}, \ldots, \lambda_n$ real eigenvalues. Now let's suppose that $\lambda_j = a_j + ib_j$ with $a, b \in \mathbb{R}$ for the complex ones ($\bar{\lambda}_j = a_j - ib_j$). Form the matrix

$$\Lambda := \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \oplus \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix} \oplus \operatorname{diag}(\lambda_{2k+1} \ldots \lambda_n).$$

Now let $\mathbf{v}_1, \bar{\mathbf{v}}_1, \ldots, \mathbf{v}_k, \bar{\mathbf{v}}_k, \mathbf{v}_{2k+1}, \ldots, \mathbf{v}_n$ be a basis of $\mathbb{R}^n$ of generalised eigenvectors in the appropriate generalised eigenspace. Then, for the complex ones, write $\mathbf{v}_j = \mathbf{u}_j + i\mathbf{w}_j$, with $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$. Form the $n \times n$ matrix

$$P := \begin{pmatrix} | & | & | & | & & | & | & | & & | \\ \mathbf{u}_1 & \mathbf{w}_1 & \mathbf{u}_2 & \mathbf{w}_2 & \cdots & \mathbf{u}_k & \mathbf{w}_k & \mathbf{v}_{2k+1} & \cdots & \mathbf{v}_n \\ | & | & | & | & & | & | & | & & | \end{pmatrix}.$$

Returning to our construction, the matrix P is invertible (because of the primary decomposition theorem). So we can make a new matrix $S = P\Lambda P^{-1}$, and a matrix $N = A - S$. We have the following:

Theorem 6.2. *Let A, N, S, Λ and P be the matrices defined above, then*

(1) $A = S + N$
(2) The matrix S is semisimple
(3) The matrices S, N and A all commute.
(4) The matrix N is nilpotent.

Proof.

(1) Follows from the definition of N.
(2) Follows from the construction of S.
(3) We show first that $[S, A] = 0$. Suppose that $\mathbf{v}$ is a generalised eigenvector of A associated to an eigenvalue λ. Then, by construction, $\mathbf{v}$ is a genuine eigenvector of S with eigenvalue λ (If $\mathbf{v}$ has a nonzero imaginary part, this needs to be split up appropriately, i.e. write $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, then $S\mathbf{u} = a\mathbf{u} - b\mathbf{w}$ where $\lambda = a + ib$, and similarly for $\mathbf{w}$). Further we note that as the generalised eigenspaces are invariant under A we have that $A\mathbf{v}$ will be a genuine eigenvector of S with eigenvalue λ too. Next apply $[S, A]$ to $\mathbf{v}$ to get $[S, A]\mathbf{v} = SA\mathbf{v} - AS\mathbf{v} = \lambda A\mathbf{v} - A\lambda\mathbf{v} = 0$. Now every element of $\mathbb{R}^n$ can be written (uniquely) as a linear combination of the $\mathbf{u}_j, \mathbf{w}_j$, and $\mathbf{v}_j$ so we can conclude that $[S, A]\mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}$. Thus $[S, A] = 0$. To see that N and S commute, observe first that $[S, N] = [S, A - S] = [S, A] = 0$ from before, and so S and N commute. Lastly, A and N commute from the definition of N and the fact that A commutes with S. This proves (3).
(4) Suppose that the maximum algebraic multiplicity of any eigenvalue of A is m. Then for any $\mathbf{v} \in E_j$ a generalised eigenspace corresponding to the eigenvalue λ_j. We have $N^m\mathbf{v} = (A - S)^m\mathbf{v} = (A - S)^{m-1}(A - \lambda_j)\mathbf{v} = (A - S)^{m-2}(A - \lambda_j)^2\mathbf{v}$ since $[S, A] = 0$, and so on and so on. So eventually we get $N^m\mathbf{v} = (A - \lambda_j)^m\mathbf{v} = 0$. Again the same argument as for (3) holds, since the E_j's span $\mathbb{R}^n$ this means that N is nilpotent.

□

Example 6.1. Compute e^{At}, and e^A when $A = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$.

The eigenvalues of A are $1, 1$. Now we find the generalised eigenspace of $\lambda = 1$. We have that $\ker(A - \mathbb{I})^2 = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2$, so we choose a basis for $\mathbb{R}^2$. I pick the standard basis. Then $P = \mathbb{I}$, and we have that $\Lambda = \mathbb{I}$ and $S = P\Lambda P^{-1} = \mathbb{I}$. Then $N = A - S = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$ is nilpotent of nilpotency 2. So we can clearly see that $A = S + N$ and that S is semisimple, while N is nilpotent and S and N commute. Now we have that $e^{At} = e^{St}e^{Nt} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}(\mathbb{I} + Nt) = e^t\begin{pmatrix} 1-2t & t \\ -4t & 2t+1 \end{pmatrix}$. Finally to get e^A just plug in $t = 1$.

Example 6.2. Compute e^{At} and e^A when

$$A = \begin{pmatrix} 5 & 1 & -1 & 1 \\ -1 & 5 & 1 & -1 \\ -1 & -1 & 3 & -1 \\ -3 & -1 & 1 & 1 \end{pmatrix}.$$

The eigenvalues of A repeated according to multiplicity are $4, 4, 4$, and 2. The generalised eigenspace for the eigenvalue $\lambda = 4$ is spanned by the vectors

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

while the eigenspace for the eigenvalue $\lambda = 2$ is spanned by the vector $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

Letting P be the matrix of eigenvalues, we have that

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

This means that $S = P\Lambda P^{-1}$ and $N = A - S$ are given by

$$S = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 4 & -2 \\ -2 & 0 & 0 & 2 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}.$$

You should verify yourself that $SN - NS = [N, S] = 0$. We have that

$$N^2 = \begin{pmatrix} -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & -2 & 2 \end{pmatrix} \quad \text{while } N^3 = 0.$$

Thus we have that $e^{At} = e^{St}e^{Nt} = Pe^{\Lambda t}P^{-1}e^{Nt}$. Multiplying these out we have

$$e^{St} = \begin{pmatrix} e^{4t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ -e^{2t}\left(-1+e^{2t}\right) & 0 & e^{4t} & -e^{2t}\left(-1+e^{2t}\right) \\ -e^{2t}\left(-1+e^{2t}\right) & 0 & 0 & e^{2t} \end{pmatrix}$$

and

$$e^{Nt} = \begin{pmatrix} -t^2+t+1 & t^2+t & t^2-t & t-t^2 \\ -t & t+1 & t & -t \\ t & -t & 1-t & t \\ t^2-t & -t^2-t & t-t^2 & t^2-t+1 \end{pmatrix}.$$

And finally putting these all together gives $e^{At} =$

$$\begin{pmatrix} e^{4t}\left(-t^2+t+1\right) & e^{4t}t(t+1) & e^{4t}(t-1)t & -e^{4t}(t-1)t \\ -e^{4t}t & e^{4t}(t+1) & e^{4t}t & -e^{4t}t \\ e^{4t}(t-1)+e^{2t} & -e^{4t}t & -e^{4t}(t-1) & e^{4t}(t-1)+e^{2t} \\ e^{4t}\left(t^2-t-1\right)+e^{2t} & -e^{4t}t(t+1) & -e^{4t}(t-1)t & e^{4t}(t-1)t+e^{2t} \end{pmatrix}.$$

To find e^A, set $t=1$ and we have

$$e^A = \begin{pmatrix} e^4 & 2e^4 & 0 & 0 \\ -e^4 & 2e^4 & e^4 & -e^4 \\ e^2 & -e^4 & 0 & e^2 \\ e^2-e^4 & -2e^4 & 0 & e^2 \end{pmatrix}.$$

Example 6.3. One last example. Compute e^{At} and e^A when

$$A = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 1 & -2 \\ -2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of A are $2 \pm 2i$, each with algebraic multiplicity 2. We have that $\ker(A-(2+2i)\mathbb{I})^2$ is spanned by the vectors

$$v_1 = \begin{pmatrix} -1-4i \\ 4+16i \\ 0 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \\ 17 \end{pmatrix} + i\begin{pmatrix} -4 \\ 16 \\ 0 \\ 0 \end{pmatrix} = u_1 + iw_1 \quad \text{and}$$

$$v_2 = \begin{pmatrix} -4-16i \\ -1-4i \\ 17 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 17 \\ 0 \end{pmatrix} + i\begin{pmatrix} -16 \\ -4 \\ 0 \\ 0 \end{pmatrix} = u_2 + iw_2$$

Thus we can write

$$P = \begin{pmatrix} -1 & -4 & -4 & -16 \\ 4 & 16 & -1 & -4 \\ 0 & 0 & 17 & 0 \\ 17 & 0 & 0 & 0 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{17} \\ -\frac{1}{68} & \frac{1}{17} & 0 & -\frac{1}{68} \\ 0 & 0 & \frac{1}{17} & 0 \\ -\frac{1}{17} & -\frac{1}{68} & -\frac{1}{68} & 0 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 2 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{pmatrix} \quad S = P\Lambda P^{-1} = \begin{pmatrix} \frac{5}{2} & 0 & 2 & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{1}{2} & -2 \\ -2 & -\frac{1}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & 2 & 0 & \frac{3}{2} \end{pmatrix}$$

and $N = A - S = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$. Then we have that $N^2 = 0$ and $SN - NS = 0$ (which you should check for yourself). This gives $e^{At} = e^{St}e^{Nt}$ where $e^{St} =$

$$e^{2t}\begin{pmatrix} \cos 2t + \frac{1}{4}\sin 2t & 0 & \sin 2t & \frac{1}{4}\sin 2t \\ 0 & \cos 2t + \frac{1}{4}\sin 2t & \frac{1}{4}\sin 2t & -\sin 2t \\ -\sin 2t & -\frac{1}{4}\sin 2t & \cos 2t - \frac{1}{4}\sin 2t & 0 \\ -\frac{1}{4}\sin 2t & \sin 2t & 0 & \cos 2t - \frac{1}{4}\sin 2t \end{pmatrix}$$

and

$$e^{Nt} = \begin{pmatrix} 1 - \frac{t}{2} & 0 & 0 & -\frac{t}{2} \\ 0 & \frac{t+2}{2} & \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} & 1 - \frac{t}{2} & 0 \\ \frac{t}{2} & 0 & 0 & \frac{t+2}{2} \end{pmatrix}.$$

And again to find e^A substitute $t = 1$.

7. Wasn't this class about ODE's?

So now that we can take the exponential of any matrix, we're ready to get to the 'point' of all this matrix exponential stuff. Consider the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x} \quad \mathbf{x}(0) = x_0, \tag{7.1}$$

where A is a real $n \times n$ matrix and $x_0 \in \mathbb{R}^n$ is a vector. This is exactly like every other initial value problem you have seen before, except now it is written in matrix form.

Theorem 7.1. *The unique solution for all $t \in \mathbb{R}$ to the initial value problem* (7.1) *is given by*

$$\boldsymbol{x}(t) = e^{At}x_0.$$

Before proving the theorem, we introduce a useful proposition.

Proposition 7.1.

$$\frac{d}{dt}\left(e^{At}\right) = Ae^{At} = e^{At}A. \tag{7.2}$$

We note that the second equality follows immediately from the series expansion of e^{At} and the fact that At and A commute for all real t. We will prove this proposition in two different ways.

Proof 1. In this proof we simply write out the series expansion for e^{At} and differentiate term by term. This gives

$$\begin{aligned}\frac{d}{dt}\left(\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \cdots\right) &= \left(A + A^2t + \frac{1}{2!}A^3t^2 + \cdots\right) \\ = A\left(\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \cdots\right) &= Ae^{At} = e^{At}A.\end{aligned}$$

□

Okay, so we can technically do this (differentiate term by term) because the exponential series converges uniformly ('recall' what this means) on any closed interval $[a, b]$ but strictly speaking it is dangerous to differentiate term by term in a series. So with that in mind we're going to present another proof which relies on the good old limit definition of the derivative.

Proof 2. Writing out the limit definition of the derivative we have

$$\begin{aligned}\frac{d}{dt}\left(e^{At}\right) &= \lim_{h\to 0}\left(\frac{e^{A(t+h)} - e^{At}}{h}\right) = \lim_{h\to 0}\left(\frac{e^{At}\left(e^{Ah} - \mathbb{I}\right)}{h}\right) \\ &= e^{At}\lim_{h\to 0}\left(\frac{e^{Ah} - \mathbb{I}}{h}\right) = \lim_{h\to 0}\frac{1}{h}\left(Ah + \frac{1}{2!}A^2h^2 + \frac{1}{3!}A^3h^3 + \cdots\right) \\ &= e^{At}\left[\lim_{h\to 0}\left(A + \frac{1}{2!}A^2h + \frac{1}{3!}A^3h^2 + \cdots\right)\right] = e^{At}A = Ae^{At}\end{aligned}$$

This completes the second proof of the proposition. □

Proof of Theorem. Now we're ready to prove the theorem. We have a function $\mathbf{x}(t) = e^{At}x_0$, and from the proposition we have that $\dot{\mathbf{x}}(t) = Ae^{At}x_0 = A\mathbf{x}(t)$, and clearly $\mathbf{x}(0) = x_0$, so we have that $\mathbf{x}(t)$ is a solution to the initial value problem for all $t \in \mathbb{R}$. Now we need to show that it is the only one. Suppose we had another solution $\mathbf{y}(t)$. Then we will consider the function $e^{-At}\mathbf{y}(t)$. We claim that this function is a constant. To see this, we have

$$\begin{aligned}\frac{d}{dt}\left(e^{-At}\mathbf{y}(t)\right) &= -Ae^{-At}\mathbf{y}(t) + e^{-At}\dot{\mathbf{y}}(t) \\ &= -Ae^{-At}\mathbf{y}(t) + e^{-At}A\mathbf{y}(t) \\ &= -e^{-At}A\mathbf{y}(t) + e^{-At}A\mathbf{y} \\ &= e^{-At}\left(-A\mathbf{y}(t) + A\mathbf{y}(t)\right) \\ &= 0.\end{aligned}$$

So we have that $e^{-At}\mathbf{y}(t)$ is a constant. Which one? Well, we just need to evaluate it at one value, so let's choose $t = 0$. Then we have $e^{-A0}\mathbf{y}(0) = x_0$. Thus we have that $e^{-At}\mathbf{y}(t) = x_0$ or $\mathbf{y}(t) = e^{At}x_0$ for all values of $t \in \mathbb{R}$. This completes the proof of the theorem. □

Now let's round things out with an example:

Example 7.1. Solve the initial value problem:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}(t) \quad \text{with } x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = 1$ with eigenvector $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We can thus use these values to compute

$$e^{At} = \frac{1}{2} \begin{pmatrix} e^t + e^{3t} & e^{3t} - e^t \\ e^{3t} - e^t & e^{3t} + e^t \end{pmatrix} = e^{2t} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

That last part isn't that important, it is just possible to deduce it from the definition of hyperbolic sine and cosine. Thus we have that the solution to the initial value problem (by the theorem) is $e^{At} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is just the first column of e^{At} or $\begin{pmatrix} e^{2t} \cosh t \\ e^{2t} \sinh t \end{pmatrix}$.

We could have also taken our initial condition as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and we would have likewise gotten the second column of e^{At}, which is just $\begin{pmatrix} e^{2t} \sinh t \\ e^{2t} \cosh t \end{pmatrix}$. We'd subsequently have a pair of linearly independent solutions, which means that by taking all linear combinations of them, we have *all* solutions to the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ (this follows from Theorem 7.1). Generalising this, if we use the standard basis vector $\mathbf{e}_k$ as our initial condition we end up with the kth column of e^{At}. If we were to do this for all of these basis vectors $\mathbf{e}_1$ through $\mathbf{e}_n$, then we'd have all the columns of e^{At}, and moreover, we'd have a full set of linearly independent solutions, and so we could, by taking linear combinations of solutions, get every solution. This discussion is summed up in the following useful corollary.

Corollary 7.1. *The set of solutions to a constant coefficient ODE of order n is an n dimensional vector space. If the coefficients are in $\mathbb{R}$, then the vector space can also be taken to be real.*

Another way to think about this is the following: e^{At} is the solution to the $n \times n$ *matrix* initial value problem

$$\dot{\Phi}(t) = A\Phi(t)$$

with the *matrix* initial condition

$$\Phi(0) = \mathbb{I}.$$

So in this way, we're able to write a full set of linearly independent solutions in one go. For this reason, the matrix e^{At} is called *the principal fundamental solution matrix* to the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ (or to the matrix ODE $\dot{\Phi} = A\Phi$).

Example 7.2. Let $V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Consider the system of ODEs

$$\ddot{\mathbf{y}} + V\mathbf{y} = \lambda \mathbf{y} \tag{7.3}$$

where $\lambda \in \mathbb{R}$.

First write the second order system of ODE's as a first order system. Let $\mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\dot{\mathbf{y}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ and rewrite eq. (7.3) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda+1 & 0 & 0 & 0 \\ 0 & \lambda-1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} =: A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{7.4}$$

We will call a solution $\mathbf{x}(t)$ to eq. (7.4) *bounded for all time* if there is an $M < \infty$ such that $||\mathbf{x}(t)|| < M$ for all $t \in \mathbb{R}$ (where by $||\mathbf{x}(t)||$ we mean the usual Euclidean norm $||\mathbf{x}(t)|| := \left(\sum_{i=1}^{4} |x_i(t)|^2 \right)^{\frac{1}{2}}$).

Question. For what values of λ do there exist solutions to eq. (7.4) which are *bounded for all time*? What is the dimension of the space of solutions which are bounded for all time for each $\lambda \in \mathbb{R}$?

(Such a value of λ if it exists is called, in an only mildly confusing abuse of language, an *eigenvalue* of the (linear) operator $\frac{d^2}{dt^2} + V$.) In an effort to incorporate everything we've looked at so far in this course, we are going to answer this question in two different ways.

Answer (First Way)**.** The first step in both ways is to compute the eigenvalues and eigenvectors of the matrix A as functions of $\lambda \in \mathbb{R}$. In order to answer this we need to compute the fundamental solution matrix for all $\lambda \in \mathbb{R}$. The eigenvalues of the matrix A are $\pm\sqrt{\lambda \pm 1}$ and the corresponding eigenvectors are

$$\mathbf{v}_{1,2} = \begin{pmatrix} \pm 1 \\ 0 \\ \sqrt{\lambda+1} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{3,4} = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \sqrt{\lambda-1} \end{pmatrix}.$$

These will be distinct provided $\lambda \neq \pm 1$. For the moment, assume that this is the case.

We then have 4 eigensolutions $\{\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t), \mathbf{y}_4(t)\}$ which are given by

$$\mathbf{y}_1(t) = e^{t\sqrt{\lambda+1}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{\lambda+1} \\ 0 \end{pmatrix}, \mathbf{y}_2(t) = e^{-t\sqrt{\lambda+1}} \begin{pmatrix} -1 \\ 0 \\ \sqrt{\lambda+1} \\ 0 \end{pmatrix},$$

$$\mathbf{y}_3(t) = e^{t\sqrt{\lambda-1}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \sqrt{\lambda-1} \end{pmatrix}, \text{ and } \mathbf{y}_4(t) = e^{-t\sqrt{\lambda-1}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ \sqrt{\lambda-1} \\ 0 \end{pmatrix}.$$

Since $\mathbf{v}_j$ are all eigenvectors with distinct eigenvalues, Proposition 4.1 says that these vectors are all linearly independent. We claim that this means that the $\mathbf{y}_i(s)$ are linearly independent as well. To see this, observe that the hypothesis of Theorem 4.1

is satisfied, and so the $\mathbf{v}_i(t)$ are the eigenvectors of e^{At} with eigenvalues $e^{\pm t\sqrt{\lambda \pm 1}}$. Now, Corollary 7.1 says that the solution space of eq. (7.4) and equivalently eq. (7.3) is a vector space of dimension four, and we have just shown that we have four linearly independent solutions to eq. (7.4). Thus we must have a basis. So, we can write *any* solution as a linear combination

$$\mathbf{x}_g(t) = k_1\mathbf{y}_1(t) + k_2\mathbf{y}_2(t) + k_3\mathbf{y}_3(t) + k_4\mathbf{y}_4(t).$$

Thus we will have a solution which is bounded for all time *precicely* when one or more of the $\mathbf{y}_i(t)'s$ is bounded for all time. This will be when any of the coefficients of t in the exponents in the eigensolutions has a zero real part. Since we are assuming that $\lambda \in \mathbb{R}$, and $\lambda \neq \pm 1$, this will be when $\lambda \in (-\infty, -1) \cup (-1, 1)$. That is $\lambda \leq 1$ (but not equal to ± 1). (Notice that when $\lambda > 1$, none of the eigensolutions are bounded and so no solution can be). Now what about when $\lambda = \pm 1$? Well, if $\lambda = -1$, then solutions $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are no longer linearly independent. However the eigensolutions $\mathbf{y}_3(t)$ and $\mathbf{y}_4(t)$ *are* still linearly independent and more over they are still bounded for all time (the coefficient of t in the exponential is purely imaginary), so when $\lambda = -1$ we have a bounded for all time solution. Now what about $\lambda = 1$?. Again, we see that $\mathbf{y}_3(t)$ and $\mathbf{y}_4(t)$ are no longer linearly independent solutions, however, they will still be bounded for all time, so again, we include $\lambda = 1$. Thus the final answer is $\lambda \in (-\infty, 1]$.

Answer (SecondWay)**.** Now that we know the answer, lets see how the exponential of a matrix sheds some light on the problem. Again, we have that the eigenvalues of the matrix A are $\pm\sqrt{\lambda \pm 1}$ and the corresponding eigenvectors are

$$\mathbf{v}_{1,2} = \begin{pmatrix} \pm 1 \\ 0 \\ \sqrt{\lambda + 1} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{3,4} = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \\ \sqrt{\lambda - 1} \end{pmatrix}.$$

These will be distinct provided $\lambda \neq \pm 1$. For the moment again, assume that this is the case. For notational convenience, we let $\mu_\pm = \sqrt{\lambda \pm 1}$. Then we have that P, the matrix of eigenvectors is given by

$$P = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \mu_+ & \mu_+ & 0 & 0 \\ 0 & 0 & \mu_- & \mu_- \end{pmatrix} \quad \text{with} \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \frac{1}{\mu_+} & 0 \\ -1 & 0 & \frac{1}{\mu_+} & 0 \\ 0 & -1 & 0 & \frac{1}{\mu_-} \\ 0 & 1 & 0 & \frac{1}{\mu_-} \end{pmatrix}.$$

As per usual, we let $\Lambda = \operatorname{diag}(\mu_+, -\mu_+, \mu_-, -\mu_-)$, and then we have that the fundamental solution matrix to eq. (7.4) is:

$$e^{At} = \begin{pmatrix} \cosh\mu_+ t & 0 & \frac{\sinh\mu_+ t}{\mu_+} & 0 \\ 0 & \cosh\mu_- t & 0 & \frac{\sinh\mu_- t}{\mu_-} \\ \mu_+\sinh\mu_+ t & 0 & \cosh\mu_+ t & 0 \\ 0 & \mu_-\sinh\mu_- t & 0 & \cosh\mu_- t \end{pmatrix}.$$

Now we have a full set of linearly independent solutions to eq. (7.4) (and eq. (7.3)). If $-1 < \lambda < 1$, then $\mu_- = \sqrt{\lambda - 1}$ will be purely imaginary, while $\mu_+ = \sqrt{\lambda + 1}$ will be real. And if $\lambda < -1$ then $\mu_\pm$ are both purely imaginary. Further, we can use the identities $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$ for $x \in \mathbb{R}$. So if $-1 < \lambda < 1$, then for notational convenience set $\nu_- := \sqrt{-(\lambda-1)}$ (note that this is real) and observe that e^{At} becomes

$$\begin{pmatrix} \cosh\mu_+ t & 0 & \frac{\sinh\mu_+ t}{\mu_+} & 0 \\ 0 & \cos\nu_- t & 0 & -\frac{\sin\nu_- t}{\nu_-} \\ \mu_+\sinh\mu_+ t & 0 & \cosh\mu_+ t & 0 \\ 0 & \nu_-\sin\nu_- t & 0 & \cos\nu_- t \end{pmatrix}.$$

So if $-1 < \lambda < 1$ we can see that two of the columns of the fundamental solution matrix are bounded for all time, so in particular, any linear combination of them are, and we can deduce that there is a two dimensional subspace of solutions to eq. (7.4) (and hence eq. (7.3)) which are bounded for all time if $-1 < \lambda < 1$.

If $\lambda < -1$, then (again for notational convenience only), let's set $\nu_\pm = \sqrt{-(\lambda \pm 1)}$. We note that now, both of these are real. Using the aforementioned relations between hyperbolic sine and cosine and their trigonometric counterparts e^{At} becomes

$$\begin{pmatrix} \cos\nu_+ t & 0 & -\frac{\sin\nu_+ t}{\nu_+} & 0 \\ 0 & \cos\nu_- t & 0 & -\frac{\sin\nu_- t}{\nu_-} \\ -\nu_+\sin\nu_+ t & 0 & \cos\nu_+ t & 0 \\ 0 & -\nu_-\sin\nu_- t & 0 & \cos\nu_- t \end{pmatrix}.$$

The point is that if $\lambda < -1$ then *all* of the columns in the fundamental matrix solution are bounded. So this means that if $\lambda < -1$ then *all* of the solutions to the system in (7.4) (and hence to eq. (7.3)) are bounded. What about when $\lambda = \pm 1$? Well, first off, the matrix P as it is written isn't invertible when $\lambda = \pm 1$. And moreover, in these cases, A has 0 as a double eigenvalue, and will be deficient. When $\lambda = 1$, we will have that

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has eigenvalues $\pm\sqrt{2}, 0, 0$. The matrix of (now generalised) eigenvectors is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\Lambda = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{so} \quad S = P\Lambda P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and we have that

$$N = A - S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad SN - SN = 0 \quad \text{and} \quad N^2 = 0.$$

So we can write $e^{At} = e^{St}e^{Nt}$ and we get

$$e^{At} = \begin{pmatrix} \cosh\sqrt{2}t & 0 & \frac{\sinh\sqrt{2}t}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{2}\sinh\sqrt{2}t & 0 & \cosh\sqrt{2}t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And putting it all together we get that when $\lambda = 1$

$$e^{At} = \begin{pmatrix} \cosh\sqrt{2}t & 0 & \frac{\sinh\sqrt{2}t}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & t \\ \sqrt{2}\sinh\sqrt{2}t & 0 & \cosh\sqrt{2}t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Examining the columns we see that one of the columns is bounded, while three are not. Thus we have that there is a one dimensional subspace of solutions which are bounded for all time.

When $\lambda = -1$, the computation is pretty much the same, we have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix},$$

which has eigenvalues $0, 0, \pm i\sqrt{2}$. The matrix of (now generalised) eigenvectors is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & -\sqrt{2} & 0 \end{pmatrix} \quad \text{so} \quad S = P\Lambda P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

And we have that

$$N = A - S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad SN - SN = 0 \quad \text{and} \quad N^2 = 0.$$

So we can write $e^{At} = e^{St}e^{Nt}$ and we get

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\sqrt{2}t & 0 & \frac{\sin\sqrt{2}t}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & -\sqrt{2}\sin\sqrt{2}t & 0 & \cos\sqrt{2}t \end{pmatrix} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And putting it all together we get that when $\lambda = -1$

$$e^{At} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & \cos\sqrt{2}t & 0 & \frac{\sin\sqrt{2}t}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & -\sqrt{2}\sin\sqrt{2}t & 0 & \cos\sqrt{2}t \end{pmatrix}.$$

Examining the columns we see that three of the columns are bounded, while one is not. Thus we have that there is a three dimensional subspace of solutions which are bounded for all time, and a one dimensional subspace which grows linearly. Putting this all together, we get the same answer as in the first way (whew), namely, there exists a bounded-for-all-time solution precisely when $\lambda \in (-\infty, 1]$.

That pretty much wraps up the quantitative study of linear constant coefficient homogeneous first order ODE's. They are all pretty much solvable (modulo computation of the exponential) and as such they are essentially trivial. Now we are going to *qualitatively* study their behaviour.

8. Stability - the Beginnings

When we consider the constant coefficient homogenous linear system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{8.1}$$

We want to answer the question "What does it mean for this linear system to be stable?" There are a couple of answers to this. The easiest and most straightforward to compute is the following:

Definition 8.1. We say that a constant coefficient, homogeneous linear system eq. (8.1) (or the critical point at the origin) is *spectrally stable* provided that none of the eigenvalues of A have positive real part. Otherwise we say it is *spectrally unstable.*

A word about the nomenclature:

Definition 8.2. The *spectrum* of a matrix A is the set of eigenvalues. We denote it by $\sigma(A)$.

Remark. We remark that if eq. (8.1) is spectrally stable, then none of the solutions to that system will grow exponentially. Further *if A is diagonalizable* (and as you well know by now, this is a *big* 'if'), with eigenvalues $\lambda_1, \dots \lambda_j$, and eigenvectors, $\mathbf{v}_1, \dots, \mathbf{v}_n$, then you know that all solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ are of the form

$$\mathbf{x}(t) = \sum_{j=1}^{n} k_j e^{\lambda_j} \mathbf{v}_j$$

and we can see that $|\mathbf{x}(t)| \leq |\mathbf{x}(0)|$ for all $t > 0$. So the solution will stay 'near' to where it starts.

We now want to put the primary decomposition theorem in this context. Let λ_j be the eigenvalues of the matrix A, write out the (possibly complex) associated generalised eigenvectors $\mathbf{v}_j = \mathbf{u}_j + i\mathbf{w}_j$. Then consider the spaces:

$$\begin{aligned} \mathbb{E}^u &= \{\mathbf{u}_j, \mathbf{w}_j | \mathrm{Re}\,(\lambda_j) > 0\} \\ \mathbb{E}^c &= \{\mathbf{u}_j, \mathbf{w}_j | \mathrm{Re}\,(\lambda_j) = 0\} \\ \mathbb{E}^s &= \{\mathbf{u}_j, \mathbf{w}_j | \mathrm{Re}\,(\lambda_j) < 0\} \end{aligned}$$

$\mathbb{E}^u$ is called *the unstable subspace* of the linear system, $\mathbb{E}^c$ is called *the centre subspace* of the linear system, and $\mathbb{E}^s$ is called *the stable subspace* of the linear system. We also will refer to the associated eigenvalues λ_j as the *unstable, centre* or *stable eigenvalues* (respectively). Because of the primary decomposition theorem we have that

$$\mathbb{R}^n = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$$

and moreover we have that each piece is invariant with respect to A, e^A and e^{At}. That is, $A : \mathbb{E}^{u,c,s} \to \mathbb{E}^{u,c,s}$ respectively, and the same for e^A, and e^{At}.

We also note that if the system in eq. (8.1) is spectrally stable, then we must have $\mathbb{E}^u = \{0\}$. We note here the following definition

Definition 8.3. We say that the system in eq. (8.1) (or the critical point at the origin) is *hyperbolic* if $\mathbb{E}^c = \{0\}$.

Evidently there is not a special term for what happens when $\mathbb{E}^s = \{0\}$.

Example 8.1 (Sort of where the term hyperbolic comes from)**.** In this example, we are going to apply what we know so far to a computationally straightforward example as well as get our feet wet with regard to some of the ideas of this section. Consider the 2×2 first order linear system $\dot{\mathbf{x}} = A\mathbf{x}$ given below

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{8.2}$$

You can verify for yourself that the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We remark that this constant coefficient homogeneous linear system is hyperbolic, but is not spectrally stable. Further we have that

$$\mathbb{E}^u = \text{Span}\{\mathbf{v}_2\}, \quad \mathbb{E}^c = \{0\}, \text{ and } \quad \mathbb{E}^s = \text{Span}\{\mathbf{v}_1\}.$$

You can also check yourself that

$$e^{At} = e^{-t} \begin{pmatrix} \cosh 2t & \sinh 2t \\ \sinh 2t & \cosh 2t \end{pmatrix},$$

and so all solutions to eq. (8.2) are of the form $e^{At}\mathbf{x}_0$ where $\mathbf{x}_0$ is the vector of initial conditions. We also know that $\mathbb{E}^s$ (and $\mathbb{E}^u$) is invariant under the dynamics of the linear system. This means that there is a (one dimensional) *vector space* of solutions of the form $ke^{-3t}\mathbf{v}_1 = ke^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -ke^{-3t} \\ ke^{-3t} \end{pmatrix}$ where $k \in \mathbb{R}$ (this idea *totally* generalises by the way, and is the topic of the next section).

Plotting these solutions in the (x_1, x_2) plane, we can see that if our solution is of this form, they will start somewhere on the line $x_2 = -x_1$ and as t increases move in towards the origin (see fig. 2). What about the other eigensolution? That is, what about the solution $\mathbf{x}(t) = ke^t\mathbf{v}_2 = ke^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} ke^t \\ ke^t \end{pmatrix}$ where $k \in \mathbb{R}$? Here too it is easy to see that we will start on the line $x_2 = x_1$ and we will stay on that line (invariance of $\mathbb{E}^u$) and as $t \to \infty$ we will travel out away from the origin. What about the rest of the solutions? Well to get a handle what happens, let's start with the initial conditions $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We know from earlier that the solution with this initial condition is just the first column of e^{At}, $\mathbf{x}(t) = e^{-t}\begin{pmatrix}\cosh 2t\\ \sinh 2t\end{pmatrix}$. Maybe you remember the equation of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. So here, $a = b = e^{-t}$, and the solution curves satisfy the equation of a hyperbola (well...), and in fact, the solution curves in the (x_1, x_2) plane all lie on hyperbolae (sort of...)

We know that the solution curve lies on a hyperbola (sort of), but we don't know yet which direction the solution will go. Later we will have quite a powerful theorem that will tell us how to determine this, but for now, the easiest way to do this is to write out $\mathbf{x}(t) = e^{-t}\begin{pmatrix}\cosh 2t\\ \sinh 2t\end{pmatrix}$ in terms of a linear combination of the solutions $e^{-3t}\mathbf{v}_1$ and $e^{t}\mathbf{v}_2$. You can check yourself that $\mathbf{x}(t) = \frac{1}{2}e^{t}\mathbf{v}_2 - \frac{1}{2}e^{-3t}\mathbf{v}_1$. Here it is clear that as $t \to \infty$ we have that the solution will tend to $\frac{1}{2}e^{t}\mathbf{v}_2$, which is on the line spanned by $\mathbf{v}_2$ in the first quadrant of the (x_1, x_2) plane. Further it is clear that as $t \to -\infty$ the solution will tend to $\frac{1}{2}e^{-3t}\mathbf{v}_1$ which is on the line spanned by $\mathbf{v}_1$ in the fourth quadrant of the (x_1, x_2) plane. (See fig. 2.) More generally, all solutions will be of the form $c_1e^{-3t}\mathbf{v}_1 + c_2e^{t}\mathbf{v}_2$ for some constants c_1 and c_2 which depend on the initial conditions (can you write the formula for c_1 and c_2 out explicitly?) and so we can see that as $t \to \infty$ *every* solution will tend towards the line spanned by $\mathbf{v}_2$ and as $t \to -\infty$ every solution will tend towards the line spanned by $\mathbf{v}_1$ (See fig. 2.)

We will pause here to make a couple of remarks. First, in fig. 2, we were able to sketch all the solutions to this ODE in the (x_1, x_2) plane, this is called the *phase portrait* of the ODE (8.2). For a planer ODE, this gives a view of many solutions in the phase plane (the (x_1, x_2) plane) all at once along with the 'direction' that time is flowing.

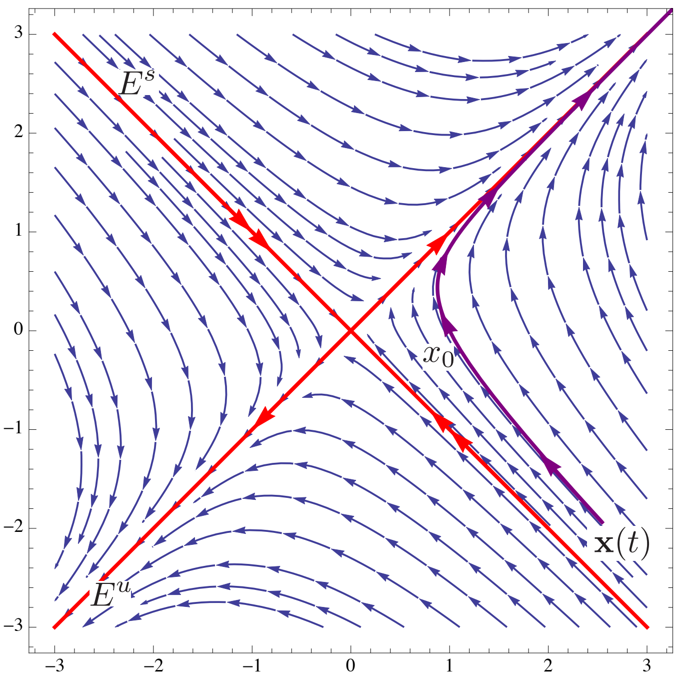


FIGURE 2. The space of solutions to the system given in eq. (8.2). The eigensolutions are in red, and the specific solution for the initial condition $\mathbf{x}_0 = (1, 0)$ is given in purple (colour online). The asymptotic behaviour of the solution is clear from the phase portrait.

Second, this system is hyperbolic, with $\mathbb{E}^u = \text{Span}\{\mathbf{v}_2\}$ and $\mathbb{E}^s = \text{Span}\{\mathbf{v}_1\}$. It can happen that either $\mathbb{E}^u$ or $\mathbb{E}^s = \{0\}$ and our system can still be hyperbolic (you just need to have $\mathbb{E}^c = \{0\}$). Next, perhaps you can see it in fig. 2 but the solutions in forward time are pulled towards $\mathbb{E}^u$ faster than the solutions are pulled towards $\mathbb{E}^s$ in backwards time. This has been reflected by putting double arrows on the stable subspace $\mathbb{E}^s$. This signifies that the solution is travelling along that subspace *faster* than it is along the subspace $\mathbb{E}^u$. This also means that strictly speaking the solutions are not travelling on hyperbolae but rather skewed hyperbolae. This can also be seen by the fact that the constants in the equation for the hyperbolae are dependent on time.

In that last example we had that the system was hyperbolic, as well as unstable. Here is an example of where the lack of hyperbolicity can result in some complicated behaviour.

Example 8.2. Let's consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where A is the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We saw earlier how $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and how $A^3 = 0$. Now the eigenvalues of A are all 0 with algebraic multiplicity 3, so in particular the linear system is spectrally stable. However if we consider the solution to the ODE with the initial condition $x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we see that $e^{At}x_0 = \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1 \end{pmatrix}$ which will grow to infinity with t (it won't grow exponentially, but it will still grow).

This motivates the following definition

Definition 8.4. We say a vector of functions $\mathbf{x}(t)$ is *bounded* if there is an $M < \infty$ such that $|\mathbf{x}(t)| \left(:= \sqrt{x_1^2(t) + \cdots + x_n^2(t)} \right) < M$ for all $t \geq 0$. Here the $x_i(t)$'s are the component functions of $\mathbf{x}(t)$.

From this we introduce the following

Definition 8.5. A constant coefficient homogeneous linear system (or the critical point at the origin) is said to be *linearly stable* if all solutions are bounded as $t \to \infty$.

So we can draw immediately the conclusions that if our linear system is hyperbolic and spectrally stable, then it is linearly stable. There is one more definition that we want to include:

Definition 8.6. A constant coefficient homogeneous linear system (or the critical point at the origin) is *asymptotically linearly stable* if $\lim_{t \to \infty} \mathbf{x}(t) = 0$ for all solutions $\mathbf{x}(t)$.

In particular this means that all of the eigenvalues of A must have negative real part. This also means that A is hyperbolic.

Remark. We note that asymptotic linear stability implies linear stability, and linear stability implies spectral stability. The reverse does not hold however. But if we add hyperbolicity, it does. That is if we know that a system is spectrally stable and hyperbolic, then it is linearly stable and asymptotically linearly stable. Likewise if a system is linearly stable and hyperbolic, then it is asymptotically linearly stable.

9. Restricted Dynamics

The idea now is to generalise what happened with $\mathbb{E}^s$, and $\mathbb{E}^u$ in Example 8.1. We saw that $E^{s,u}$ were invariant subspaces, and moreover the dynamics of the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ *restricted* to $\mathbb{E}^{s,u}$ were governed by the (lower dimensional) ODEs $\dot{v}_1 = \lambda v_1$ and $\dot{v}_2 = \lambda_2 v_2$. The key idea is that the invariance of the the subspaces $\mathbb{E}^{u,c,s}$ with respect to A and e^{At} means that we can, by mimicking the diagonalization procedure, get a good description on the dynamics of each piece. This may seem a bit disconcerting at first glance, but it is incredibly useful.

We will go through the procedure in theory for $\mathbb{E}^u$. The process is essentially the same for $\mathbb{E}^s$ and $\mathbb{E}^c$. Suppose that $\mathbf{x} \in \mathbb{E}^u$ were some vector in the unstable subspace of eq. (8.1). Then let $\mathbf{v}_1, \ldots \mathbf{v}_k$ be a basis of generalised eigenvectors for $\mathbb{E}^u$. We can write $\mathbf{x}$ uniquely as a linear combination of the $\mathbf{v}_j$s:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \quad c_i \in \mathbb{R}.$$

Now let P be the matrix whose columns are the $\mathbf{v}_j$s, $P = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & | & \end{pmatrix}$.

Then, writing $\mathbf{c} = (c_1, \ldots, c_k)^T$, we can write $\mathbf{x} = P\mathbf{c}$. Now suppose that we do this for all the columns of AP (these are all in $\mathbb{E}^u$ because of invariance). Now for each column, we have a new set of c_js and we put these into the columns of a new matrix U. Putting this all together we get

$$AP = PU,$$

where the columns of U are the coefficients corresponding to writing out the columns of AP in the basis of P.

Remark. As a quick aside, we will count the size of the matrix U. We have that A is $n \times n$, and P is $n \times k$, so AP is $n \times k$. Now we count the right hand side of $AP = PU$, we know that it must be $n \times k$ and again we know that P is $n \times k$, so U must be a $k \times k$ matrix.

Okay, so the great (intellectual) leap forward here is that we get a handle on how the dynamics of eq. (8.1) behave when restricted to the subspace $\mathbb{E}^u$ by thinking about how the elements of U change with time. We have that $\mathbf{x} = P\mathbf{c}$ and so differentiating we get

$$\dot{\mathbf{x}} = P\dot{\mathbf{c}} = A\mathbf{x} = AP\mathbf{c} = PU\mathbf{c}.$$

So we are left with

$$P\dot{\mathbf{c}} = PU\mathbf{c}.$$

Now P is a $n \times k$ matrix whose columns are linearly independent. This means that $P^\top P$ will be a $k \times k$ invertible matrix (this is because the kernel of $P^\top =$ the orthogonal complement of the image of P). Thus we can multiply both sides by $(P^\top P)^{-1}P^\top$ and get that $\dot{\mathbf{c}} = U\mathbf{c}$. So what we have just shown is the following

Theorem 9.1. *The dynamics of eq.* (8.1) *when restricted to the subspace* $\mathbb{E}^u$ *are governed by the (now* $k \times k$*) dynamical system*

$$\dot{\boldsymbol{c}} = U\boldsymbol{c}.$$

Example 9.1. This is maybe not so clear in the abstract. So let's compute an example. Let's consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where A is the matrix $A = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. We have that the eigenvalues are given as $\lambda_{1,2} = -1$ and $\lambda_3 = 1$.

Then we can also compute that the generalised eigenspaces are $\mathbb{E}^u = \text{Span}\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\} = \text{Span}\{\mathbf{v}_3\}$ and $\mathbb{E}^s = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. On $\mathbb{E}^u$ the dynamics are governed by $AP = PU$ where $P = \begin{pmatrix} | \\ \mathbf{v}_3 \\ | \end{pmatrix}$ the 3×1 matrix which is just the eigenvector $\mathbf{v}_3$. But this means that U must be the 1×1 matrix $\begin{pmatrix} 1 \end{pmatrix}$ consisting of the eigenvalue λ_3. What this means is that on the unstable subspace, the dynamics of $\dot{\mathbf{x}} = A\mathbf{x}$ are governed by the equation $\dot{\mathbf{c}} = \lambda_3 \mathbf{c} = \mathbf{c}$. This might be a bit too simple, and so misleading, so let's finish the example and compute the dynamics on the stable subspace $\mathbb{E}^s$. This time the matrix P is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. We need to solve the matrix linear equation

$$AP = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = PU.$$

It should be almost automatic to see that this means that $U = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. So on $\mathbb{E}^s$ we have that the dynamics of $\dot{\mathbf{x}} = A\mathbf{x}$ are given by $\dot{\mathbf{c}} = U\mathbf{c}$ which in the coordinates of the generalised eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ on the stable subspace takes the form:

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ -c_2 \end{pmatrix}.$$

We conclude this section with another example of this procedure, and one that also illustrates the fact that if a linear system is spectrally stable, and the dynamics restricted to $\mathbb{E}^c$ aren't deficient (the matrix U isn't deficient when considered on the central subspace) then we also have linear stability.

Example 9.2. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where the matrix A is given as

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & -4 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The eigenvalues of A (repeated according to their multiplicity) are $\{-5, -2, 0, 0\}$ with the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ (in the same order) as

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 4 \end{pmatrix}, \text{ and } \begin{pmatrix} -3 \\ -1 \\ 4 \\ 0 \end{pmatrix}.$$

Thus we have that $\mathbb{E}^u = \{0\}$, $\mathbb{E}^c = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$, and $\mathbb{E}^s = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. To understand the dynamics on $\mathbb{E}^s$, we need to solve $AP^s = P^sU^s$ where U^s is a 2×2 matrix of unknowns, and P^s is the matrix whose columns are the stable eigenvectors (the ones whose associated eigenvalue has $\text{Re}(\lambda_j) < 0$),

$$P^s = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solving $AP^s = P^sU^s$ gives (this should be obvious from mimicking the diagonalization procedure)

$$U^s = \begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix}.$$

So by applying Theorem 9.1 we have that the dynamics of $\dot{\mathbf{x}} = A\mathbf{x}$ restricted to $\mathbb{E}^s$ (in this basis of eigenvectors) are given by $\dot{\mathbf{c}} = U^s\mathbf{c}$. That these are straightforward dynamics is really the whole reason we chose this basis. Now we will examine the dynamics on $\mathbb{E}^c$ and see that we have linear stability as well. It is straightforward to see that when we solve $AP^c = P^cU^c$ as before (only this time, P^c is the 2×4 matrix whose columns are $\mathbf{v}_3$ and $\mathbf{v}_4$ or any basis for the null space of A), we have that $U^c = 0$. Thus by applying the theorem, we have that the dynamics are trivial on $\mathbb{E}^c$. Thus solutions all either decay, or stay constant (at their initial conditions), and so we have linear stability.

10. Classification of 2-D Linear Systems

> **Note:** There are some accompanying pictures to this section. Small versions are included at the end. For larger versions, you should consult the course website.

In Example 8.1, we sketched the *phase portrait* of a 2×2 first order system of equations from a qualitative perspective. We sketched the eigenvalues, and found the eigenvectors and eigensolutions, and then discussed what happens to all such solutions, sketching many of them in the phase plane. A sketched solution in this fashion is called a *phase curve*.

The idea now is to perform this analysis for a general 2×2 real constant coefficient homogeneous system of ODEs. We consider the general case of $\dot{\mathbf{x}} = A\mathbf{x}$ given:

$$(*) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

and we want to understand *all* the possible behaviours of the solutions to $(*)$ for arbitrary real constants a, b, c, d, and an arbitrary initial condition $\mathbf{x}_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}$.

It is worth noticing that if we start with $\mathbf{x}_0 = (0,0)$ then we will have the *trivial solution* $\mathbf{x}(t) = 0$. Further we have that at the point $(0,0)$, $\dot{\mathbf{x}} = A0 = 0$. This makes the origin a *critical point* of the system $(*)$. We will soon see that in 'most' cases, the origin is the *only* critical point of eq. $(*)$.

A good place to start is the characteristic equation of A. We write this out in terms of $\tau = \mathrm{tr}\,(A) = (a+d)$ and $\delta = \det\,(A) = ad - bc$. We have:

$$(\rho) \qquad \rho(\lambda) = \lambda^2 - \tau\lambda + \delta.$$

The eigenvalues of A are given by

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

Remarkably, it turns out that knowledge of only τ and δ is sufficient to qualitatively classify all the possible behaviours of 2-D linear systems. For the purposes of convenience we will introduce one more term called the *discriminant* of eqs. $(*)$ and (ρ)

$$\Delta := \tau^2 - 4\delta.$$

To organise our thoughts, we will use the (τ, δ) plane and the curve $\Delta = 0$. This together with the line $\delta = 0$ and the half-line $\tau = 0, \delta > 0$ will divide the (τ, δ) plane up into five main regions I- V, as well as five border regions (a) - (e). We'll start our classification with the 'biggest' case, the case where $\delta < 0$.

(I) $\delta < 0$: (This is a generalisation of Example 8.1.) In this case we have that both $\lambda_\pm$ are real, and moreover we have that one is positive and one is negative. That is we have that $\lambda_- < 0 < \lambda_+$. This means that we have linearly independent eigenvectors $\mathbf{v}_-$ and $\mathbf{v}_+$. Now in general we know that all solutions look like $e^{At}x_0$. However, in this case it behoves us to consider the eigensolutions we defined earlier: namely $\mathbf{x}_-(t) = e^{\lambda_- t}\mathbf{v}_-$ and $\mathbf{x}_+(t) = e^{\lambda_+ t}\mathbf{v}_+$. As in the example, these solutions are going to be *lines* in the (x_1, x_2) phase plane, and moreover $\mathbb{E}^u = \mathrm{Span}\{\mathbf{v}_+\}$ while $\mathbb{E}^s = \mathrm{Span}\{\mathbf{v}_-\}$. We also have an outward flow along $\mathbf{v}_+$ and an flow into the origin along $\mathbf{v}_-$. Rather than writing the general solution as the exponential of A, it is easier to see what is going on if we consider instead the general solution to the ODE in terms of the solutions $\mathbf{x}_-(t)$ and $\mathbf{x}_+(t)$. We can write the general solution as

$$\mathbf{x}(t) = k_1 e^{\lambda_- t}\mathbf{v}_- + k_2 e^{\lambda_+ t}\mathbf{v}_+,$$

where $k_{1,2}$ are found by solving the linear equation $\begin{pmatrix} \mathbf{v}_- & \mathbf{v}_+ \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}$. Given the form that our general solution has, we see that as $t \to \infty$ we have that $\mathbf{x}(t) \to k\mathbf{v}_+$ where k is some arbitrary constant. (We will make this more precise later.) This means that no matter what initial condition you choose, you would end up (as $t \to \infty$) getting asymptotically close to the line spanned by $\mathbf{v}_+$ in the (x_1, x_2) plane, however you can't cross the unstable and stable subspaces (why?), so whatever quadrant relative to $\mathbb{E}^u$ and $\mathbb{E}^s$ you start in, that's where you stay. This type of 2-D linear system is called a *saddle*. In this case we also say that the origin as a critical point is a saddle.

We remark here that because we are working with linear autonomous systems, the classification of the dynamics of the system is the same as describing what type of critical point the origin is, so we use the same words. We shall see that in the case of nonlinear planar systems, we will only really be able to classify the dynamics of (some of the) critical points. That takes care of the lower half of the (τ, δ) plane, now on to region II.

(II) $\tau > 0, \Delta > 0$: This corresponds to the region in the right upper half plane below the curve $\delta = \frac{\tau^2}{4}$. In this instance there are two real positive eigenvalues $0 < \lambda_- < \lambda_+$. Again we have two linearly independent eigenvectors $\mathbf{v}_+$ and $\mathbf{v}_-$ from which we can form the eigensolutions $e^{\lambda_+ t}\mathbf{v}_+$ and $e^{\lambda_- t}\mathbf{v}_-$ that span the unstable subspace (now all of $\mathbb{R}^2$). We call this system/the origin as a critical point an *unstable node* or a *nodal source.* Just like with a saddle, it's simplest to write down this solution with initial conditions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is

$$\mathbf{x}(t) = e^{At}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Also just as in the saddle case, we can write out solutions in the basis of simple solutions, so we have

$$\mathbf{x}(t) = k_1 e^{\lambda_+ t}\mathbf{v}_+ + k_2 e^{\lambda_- t}\mathbf{v}_-$$

with the k_i coming from the initial conditions as before. Now we can see that even though both of the exponents are positive, we will have that the larger one will dominate as $t \to \infty$ that is all solutions will tend to

$$\mathbf{x}(t) \to e^{\lambda_+ t}\mathbf{v}_+ + \mathbf{c}$$

where the $\mathbf{c}$ is an offset coming from the initial conditions. (Why doesn't this term appear in the saddle case?)

(III) $\tau > 0, \Delta < 0$: Here, we have two complex conjugate eigenvalues $\lambda_\pm = \frac{1}{2}(\tau \pm i\sqrt{-\Delta})$, but both with positive real part. We also have two complex conjugate eigenvectors $\mathbf{v}_\pm$. Since they are complex, we don't have simple real solutions this time. This case is called an *unstable focus* or a *spiral source.* To better understand what's going on, we let $P = [\mathbf{u}, \mathbf{w}]$ be the matrix consisting of the real and imaginary parts of the complex conjugate eigenvectors $\mathbf{v}_\pm$. We can then put all that we've done so far together to get that our solution will be of the form

$$\mathbf{x}(t) = e^{\frac{\tau}{2}t}P\begin{pmatrix} \cos\frac{t}{2}\sqrt{-\Delta} & \sin\frac{t}{2}\sqrt{-\Delta} \\ -\sin\frac{t}{2}\sqrt{-\Delta} & \cos\frac{t}{2}\sqrt{-\Delta} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Where the y_is satisfy $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^{-1}\mathbf{x}_0$. As $t \to \infty$ we have that this is going to grow exponentially, while simultaneously spiraling around the origin of the (x_1, x_2) plane. The question is which direction does it spin? Counterclockwise or clockwise? One way to do this is to just pick the initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and plug it into your original ODE. That is, compute $\dot{\mathbf{x}} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This arrow will have some vertical component (it has to - why?) if that vertical component is positive, then the arrow is pointing up and we have a counter clockwise motion. If the vertical component is negative, then the arrow is pointing downwards, and we have a clockwise motion. One nice thing to note here is that $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$ so we can conclude that if $c > 0$, we have counter clockwise motion, while if $c < 0$, we'll have clockwise motion. (Think about why $c \neq 0$.)

Question. There is another way to figure out, which direction the focus is turning, from the general solution. Can you determine it?

(IV) $\tau < 0, \Delta < 0$ This case is identical to the previous one except that now our solutions all spiral in to the origin. We have two complex conjugate eigenvalues $\lambda_\pm = \frac{1}{2}(\tau \pm i\sqrt{-\Delta})$, and two linearly independent eigenvectors $\mathbf{v}_\pm$ which are complex conjugates of each other. Again, letting $P = [\mathbf{u}, \mathbf{w}]$ we write our solutions as

$$\mathbf{x}(t) = e^{\frac{\tau}{2}t}P\begin{pmatrix}\cos\frac{t}{2}\sqrt{-\Delta} & \sin\frac{t}{2}\sqrt{-\Delta}\\ -\sin\frac{t}{2}\sqrt{-\Delta} & \cos\frac{t}{2}\sqrt{-\Delta}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}.$$

And we see that our solutions will spiral into the origin around ellipses (either clockwise or counter clockwise again determined by the same reasoning). This is called a *stable focus* or *spiral sink*.

(V) $\tau < 0, \Delta > 0$ Here, we have a case similar to (3), except that we have two real negative eigenvalues $\lambda_- < \lambda_+ < 0$. This won't change the form of our general solution, either

$$e^{At}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

or

$$\mathbf{x}(t) = k_1\mathbf{v}_+e^{\lambda_+ t} + k_2\mathbf{v}_-e^{\lambda_- t},$$

with the x_i's being the initial conditions, and the k_i's coming from them as before. Now however, as all the eigenvalues are negative, our solutions will all tend to 0 as $t \to \infty$. However, they will do so along the 'most negative' eigenvector. This is called a *stable node* or a *nodal sink*

That takes care of the five main regions. We next consider the borders between the regions. For the most part a good way to see what happens where is to consider how to turn one of the larger regions into its neighbour - for instance, consider how one might pass from region I to region II. In this case you would have the negative eigenvalue in region I would become the smaller of the positive eigenvalues in region II. But this means that on the border of the two, you would have a zero eigenvalue. So we need to investigate qualitatively (i.e. think about) what that means. This type of reasoning can/will be extended to each of the five border cases.

(a) In this case we have $\tau > 0$ and $\delta = 0$. Here, as we stated above, the eigenvalues satisfy $0 = \lambda_- < \lambda_+$, so we have one positive eigenvalue and one zero eigenvalue. We do still have two linearly independent eigenvectors and so we have two 'special' solutions like before. This time though one of them ($\mathbf{v}_-$) is constant. This means that there is no motion in the $\mathbf{v}_-$ direction. We write the general solution out as

$$\mathbf{x}(t) = k_1e^{\lambda_+ t}\mathbf{v}_+ + k_2\mathbf{v}_-.$$

We can see that even though there is no motion on $\mathbf{v}_-$ if we start anywhere off it we will travel on the line parallel to $\mathbf{v}_+$ passing through the initial condition.

This is called a *degenerate* or *non isolated unstable equilibrium.* It is worth noting here that we actually have a whole subspace of critical points spanned by $\mathbf{v}_-$ (including the origin). Each one of them is called a degenerate equilibrium, and the system is called this as well in this instance.

(b) $\tau > 0, \Delta = 0$: Here we have a repeated, positive eigenvalue $\lambda = \dfrac{\tau}{2}$ of algebraic multiplicity 2. We know that the geometric multiplicity must be either one or two. For the time being, we will assume that the geometric multiplicity is 1 as the case of geometric multiplicity equal to 2 is not really a 2-D linear system (see special case (**)). So we only have one eigenvector $\mathbf{v}$, and our general solution takes the form (after some calculations):

$$\mathbf{x}(t) = e^{At}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\lambda t}(\mathbb{I} + (A - \lambda\mathbb{I})t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We also have a single eigensolution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ and a single line in the (x_1, x_2) plane which remains invariant. This type of system is called an *unstable improper node.*

(c) $\tau = 0, \delta > 0$. In this case we have two complex conjugate, purely imaginary eigenvalues $\lambda_\pm = \pm i\sqrt{\delta}$, and two complex conjugate eigenvectors $\mathbf{v}_\pm = \mathbf{u} \pm i\mathbf{w}$. Again we form the matrix $P = [\mathbf{u}, \mathbf{w}]$ and we have that solutions will be of the form

$$\mathbf{x}(t) = P\begin{pmatrix} \cos t\sqrt{\delta} & \sin t\sqrt{\delta} \\ -\sin t\sqrt{\delta} & \cos t\sqrt{\delta} \end{pmatrix}\begin{pmatrix} y_1 \\ y_1 \end{pmatrix}.$$

These solutions will travel along ellipses themselves. That is the orbits will close up. Determination of clockwise orbit or counterclockwise orbit will be the same as in the focus cases. This is called a *center.*

(d) $\tau < 0, \Delta = 0$: This is the same as (b), only now all of the solutions will tend to 0. That is we make the same analysis as in the (b) case, only for our qualitative picture, we reverse the arrows. We have a single eigenvalue λ and a single eigenvector $\mathbf{v}$ netting a single eigensolution $e^{\lambda t}\mathbf{v}$ and our general solution will have the same form $e^{At}x_0 = e^{\lambda t}(\mathbb{I} + (A - \lambda\mathbb{I})t)x_0$. This is called a *stable improper node.*

(e) $\tau < 0, \delta = 0$: Here we have a similar situation as to case (a), only this time we don't have any motion along the vector $\mathbf{v}_+$. Further we have that our solutions will have the form

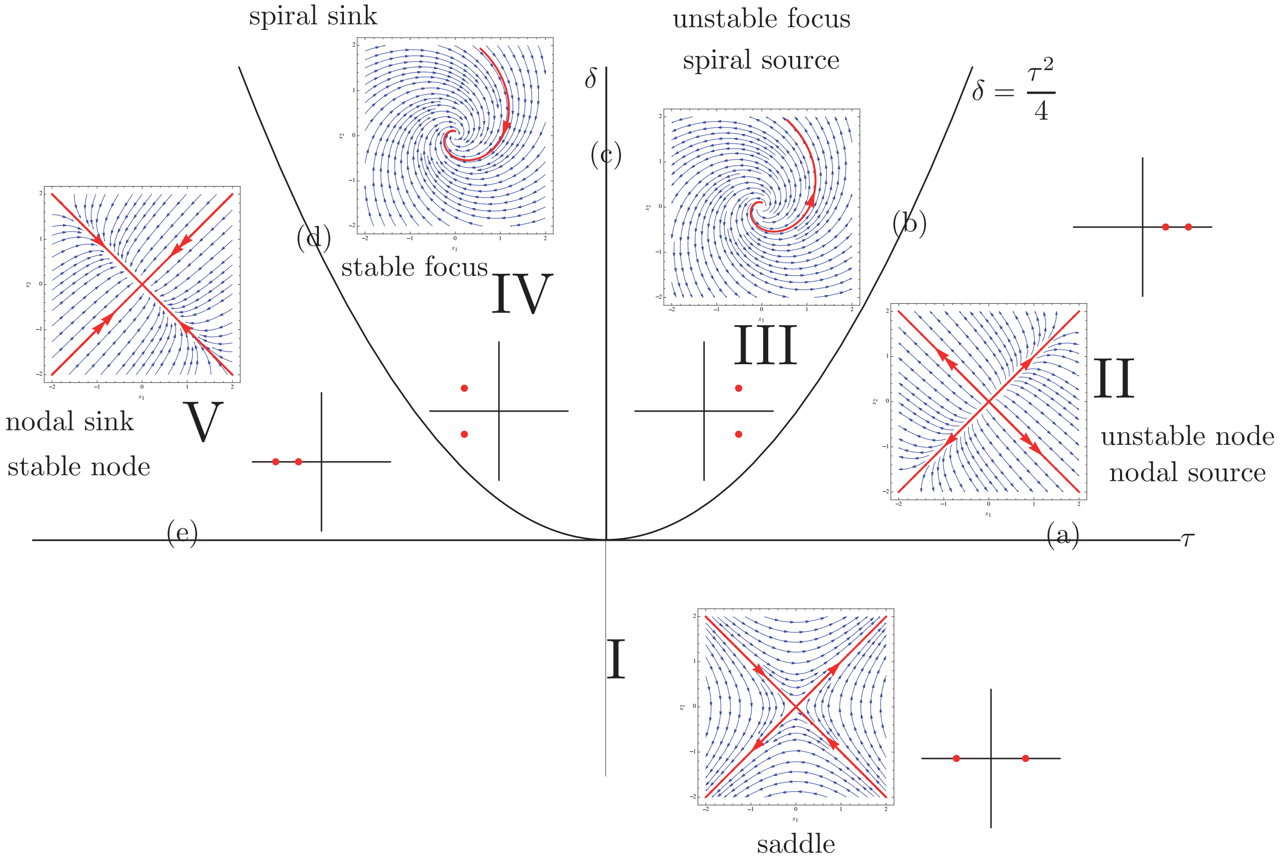
$$\mathbf{x}(t) = k_1 e^{\lambda_- t}\mathbf{v}_- + k_2\mathbf{v}_+$$

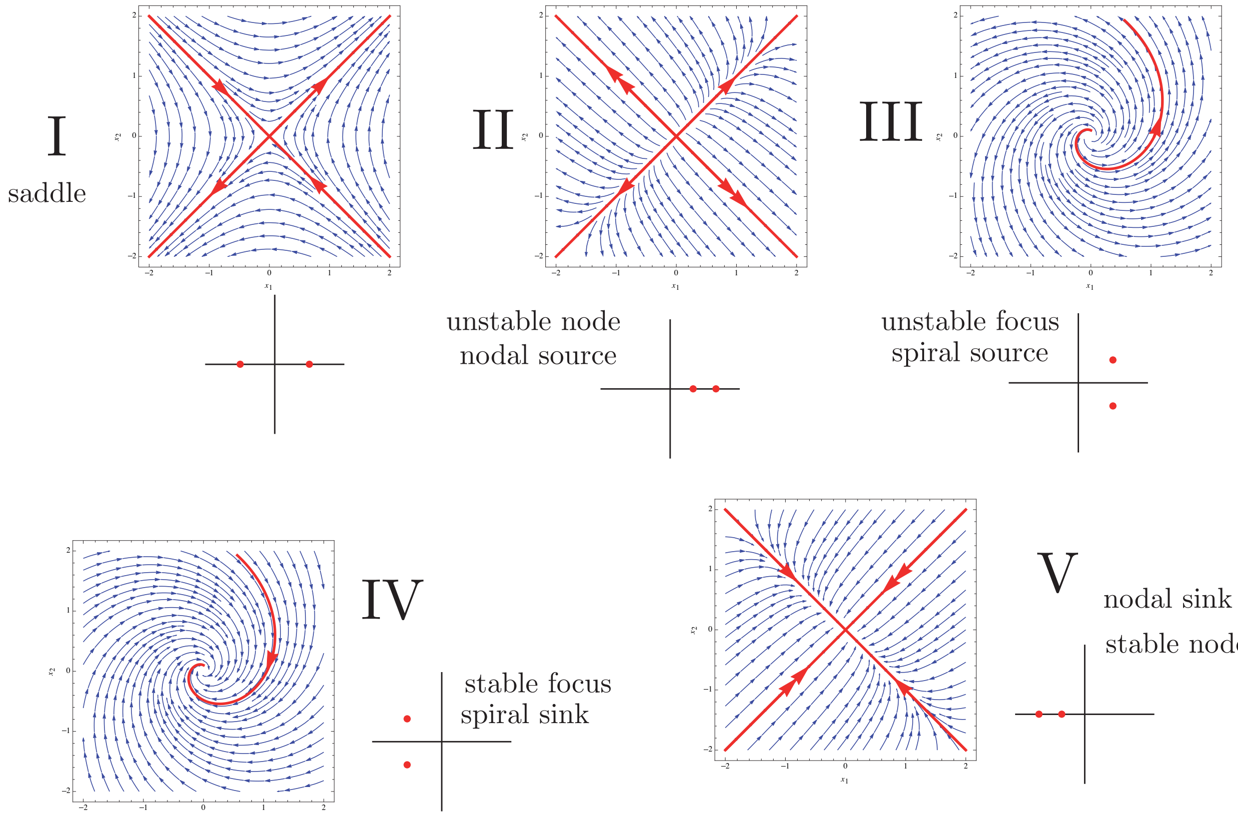
and this means that all the solutions will tend into the line $\mathbf{v}_+$. This type of system (and the critical points along $\mathbf{v}_+$) is called a *degenerate stable equilibrium.*

Special cases While the above takes care of almost every single case, and certainly the 'majority' of them, there are two exceptional cases that need to be addressed. The first deals with the last, unclaimed point in the (τ, δ) plane, while the second deals with the case of a repeated eigenvalue with geometric multiplicity 2.

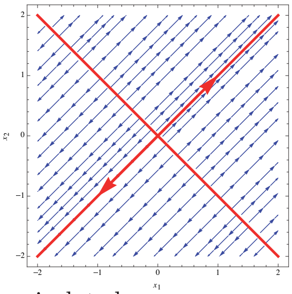
(*) $\tau = \delta = 0$ In this case we have a single repeated eigenvalue of 0, and only a single eigenvector. This means that A is nilpotent, of nilpotency 2. We have that (from the tutorials) A must have the form $A = \begin{pmatrix} a & b \\ \frac{-a^2}{b} & -a \end{pmatrix}$. We must have that $b \neq 0$ because otherwise then $A = 0$ the zero matrix and we'd have two linearly independent eigenvectors. The form of A tells us that we have one eigenvector $\mathbf{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$ and the other generalised eigenvector can be chosen to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this basis A has the form $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ so we have a very simple dynamics: along $\mathbf{v}$ we have no flow, but for every line parallel to $\mathbf{v}$ we have flow parallel to $\mathbf{v}$. I needed a name for this exceptional case, as all the other ones had names, so I called it *nilpotent flow.* Full disclosure though, this is non-standard, but it should be reasonably clear where the name comes from.

(**) Now we deal with the case of when we have a repeated eigenvalue but two, linearly independent eigenvectors. In this instance $A = \text{diag}(\lambda, \lambda)$, A is a diagonal matrix, and the system *decouples.* This means it splits into two one dimensional linear ODE's and flow in this case is simply radially out or in along the ray from the origin to the initial condition depending on the sign of λ.

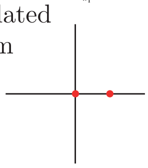

spiral sink
unstable focus
spiral source
δ
$\delta = \frac{\tau^2}{4}$
(c)
(d)
(b)
stable focus
IV
III
II
V
nodal sink
stable node
unstable node
nodal source
(e)
(a)
τ
I
saddle


I
saddle
II
III
unstable node
nodal source
unstable focus
spiral source
IV
stable focus
spiral sink
V
nodal sink
stable node

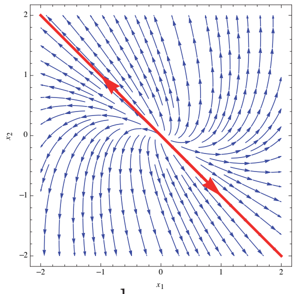
(a)



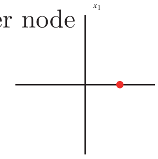
degenerate/non-isolated
unstable equilibrium



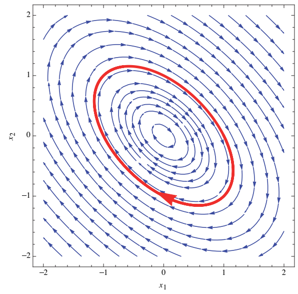
(b)



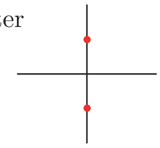
unstable improper node



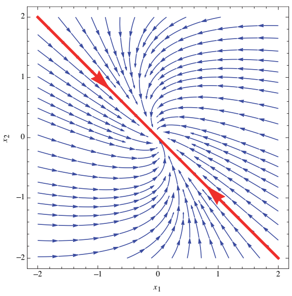
(c)



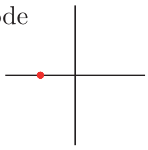
center



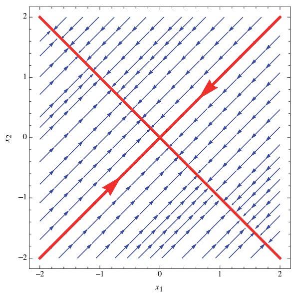
(d)



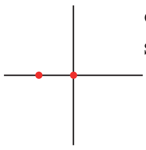
stable improper node

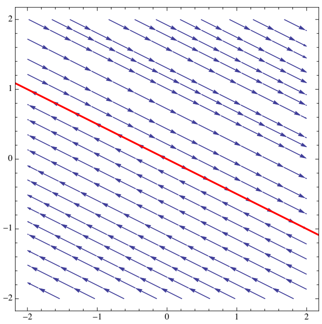


(e)



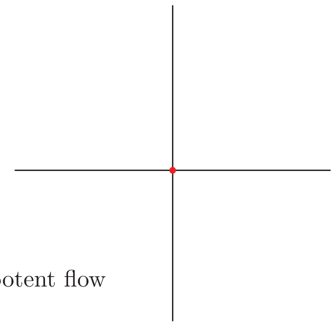
degenerate/non-isolated
stable equilibrium

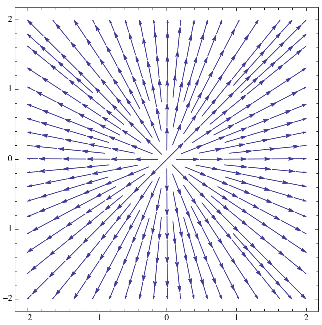




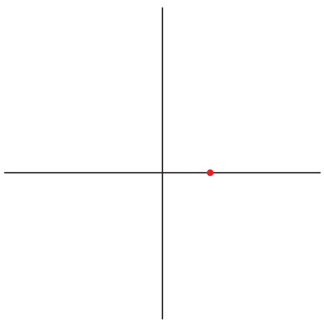
$(*)$

nilpotent flow





$(**)$



decoupled

11. Systems with Non-constant Coefficients

Now we consider the case where we have a linear first order system of equations still but the matrix A is an $n \times n$ matrix of *functions*. That is, rather than consider

$$\dot{\mathbf{x}} = A\mathbf{x} \quad \mathbf{x}(0) = x_0 \tag{11.1}$$

where A is a matrix of constants, we consider the IVP

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \quad \mathbf{x}(t_0) = x_0. \tag{11.2}$$

Besides changing our equation so that it is no longer a constant coefficient equation, we have also changed our initial condition. Rather than starting at $t = 0$ we want to consider systems which start at an arbitrary point t_0. One reason for this is that $A(t)$ may not be defined on all of $\mathbb{R}$. This happens for scalar valued ODEs as well, just consider the ODE

$$\dot{x} = \frac{1}{t}x$$

where $x(t)$ is a function from $\mathbb{R} \to \mathbb{R}$. The right hand side of the above equation isn't defined for $t = 0$, so any solution needs to avoid that point in the domain as well. In our study of constant coefficient ODEs we found the *principal fundamental matrix solution (PFSM)* at $t_0 = 0$ to be e^{At}. So if we wanted to slightly generalise this, we have that the PFSM at t_0 of the ODE in eq. (11.1) is

$$\exp(A(t - t_0)) = \exp\left(\int_{t_0}^{t} A\, ds\right).$$

The columns of this matrix are a linearly independent set of n vectors of functions and the matrix satisfies the matrix initial value problem

$$\dot{\Phi} = A\Phi \quad \Phi(t_0) = \mathbb{I}. \tag{11.3}$$

This last sentence is in effect the definition of a PFSM.

Definition 11.1. A *fundamental solution matrix* to $\dot{\mathbf{x}} = A(t)\mathbf{x}$, where $A(t)$ is an $n \times n$ matrix is a matrix solution to $\dot{\Phi}(t) = A(t)\Phi(t)$ with n linearly independent columns. If $\Psi(t)$ is a fundamental solution matrix and $\Psi(t_0) = \mathbb{I}$ for $t_0 \in \mathbb{R}$, we say that $\Psi(t)$ is a *principal fundamental solution matrix* at t_0.

We would like to find the fundamental solution matrix for eq. (11.2). It turns out that this is not so easy. In particular, it is *not* the case that the PFSM of eq. (11.2) is $\exp\left(\int_{t_0}^{t} A(s)ds\right)$, in fact, this matrix won't even solve the ODE in general. The reason for this is because the matrices $A(t)$ and $B(t) := \int_{t_0}^{t} A(s)ds$ do not in general commute.

Example 11.1. Consider the ODE

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & -\frac{1}{t} \\ 1 + t & -1 \end{pmatrix} \mathbf{x} \qquad t > 0. \tag{11.4}$$

First notice that $A(t) := \begin{pmatrix} 1 & -\frac{1}{t} \\ 1+t & -1 \end{pmatrix}$ is not defined for $t = 0$, so any solution can not include 0 as part of its domain. You can verify yourself that

$$\Phi(t) := \begin{pmatrix} 1 & -\log(t) \\ t & 1 - t\log(t) \end{pmatrix}$$

is a FSM. In order to write down a PFSM, we need to choose a t_0. Let's choose $t_0 = 1$. You can see that $\Phi(1) := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and so we have that

$$\Psi(t) := \Phi(t)\Phi^{-1}(1)$$

is a principal fundamental solution matrix. This means that any solution to the initial value problem $\dot{\mathbf{x}} = A(t)\mathbf{x}$ with $\mathbf{x}(1) = x_0$ is given by $\Psi(t)x_0$. The question is does

$$\Psi(t) = \exp\left(\int_1^t A(s)ds\right)?$$

The answer is no. This can be verified by computing the matrix exponential and then evaluating at $t = 2$. We have that $\exp\left(\int_1^2 A(s)ds\right) = \begin{pmatrix} 1.53 & * \\ * & * \end{pmatrix}$ while $\Psi(2) = \begin{pmatrix} 1.69 & * \\ * & * \end{pmatrix}$ so these matrices can not be equal. Moreover, $\exp\left(\int_1^2 A(s)ds\right)$ doesn't even solve the ODE! The best way I know to show this is by using Mathematica. For convenience, denote $B(t) := \int_1^t A(s)ds$. We have that $\frac{d}{dt}\exp(B(t))\big|_{t=2} = \begin{pmatrix} 0.09 & * \\ * & * \end{pmatrix}$, while $A(2)\exp(B(2)) = \begin{pmatrix} 0.008 & * \\ * & * \end{pmatrix}$.

There is not a general way to solve $\dot{\mathbf{x}} = A(t)\mathbf{x}$ when $A(t)$ is not a constant matrix, but all is not lost. We have the following extremely useful theorem

Theorem 11.1 (Liouville's /Abel's formula)**.** *Let $\Phi(t)$ be a fundamental matrix solution to eq.* (11.2)*. Then*

$$\det \Phi(t) = \det \Phi(t_0) e^{\int_{t_0} tr(A(s))ds}.$$

The proof for the general case is a bit technical, so we are omitting it. A tutorial question asks you to tackle the case when $A(t)$ is a 2×2 matrix. The main idea of the proof is to show that $\det(\Phi(t))$ satisfies the following (scalar) differential equation

$$\det(\Phi(t))' = \operatorname{tr}(A(t))\det(\Phi(t))$$

and the result follows.

