Exponents and Logarithms: Applications and Calculus

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Jackie Nicholas
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1 Introduction

The exponential and logarithmic functions are important functions in science, engineering and economics. They are particularly useful in modelling mathematically how populations grow or decline. You might be surprised to learn that scales used to describe the magnitude of seismic events (the Richter scale) or noise (decibels) are logarithmic scales of intensity.

In this booklet we will demonstrate how logarithmic functions can be used to linearise certain functions, discuss the calculus of the exponential and logarithmic functions and give some useful applications of them.

If you need a detailed discussion of index and log laws, then the Mathematics Learning Centre booklet: *Introduction to Exponents and Logarithms* is the place to start. If you are unsure of the level you need, then do this short quiz. The answers are in section 5.

1.1 Exercises

The following expressions evaluate to quite a 'simple' number. If you leave some of your answers in fractional form you won't need a calculator.

1. $9^{\frac{1}{2}}$ **2.** $16^{\frac{3}{4}}$ **3.** $(\frac{1}{5})^{-2}$ **4.** $(3^{-1})^2$ **5.** $(\frac{5}{2})^{-2}$

6. $(-8)^{\frac{3}{2}}$ **7.** $(-\frac{27}{8})^{\frac{2}{3}}$ **8.** $5^{27}5^{-24}$ **9.** $8^{\frac{1}{2}}2^{\frac{1}{2}}$ **10.** $(-125)^{\frac{2}{3}}$

These look a little complicated but are equivalent to simpler ones. 'Simplify' them. Again, you won't need a calculator.

11. $\dfrac{3^{n+2}}{3^{n-2}}$ **12.** $\sqrt{\dfrac{16}{x^6}}$ **13.** $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$

14. $(x^2+y^2)^{\frac{1}{2}}-x^2(x^2+y^2)^{-\frac{1}{2}}$ **15.** $\dfrac{x^{\frac{1}{2}}+x}{x^{\frac{1}{2}}}$ **16.** $(u^{\frac{1}{3}}-v^{\frac{1}{3}})(u^{\frac{2}{3}}+(uv)^{\frac{1}{3}}+v^{\frac{2}{3}})$

17. Sketch the graphs of the functions $f(x)=3^x$ and $f(x)=3^{-x}$. On the same diagrams mark in roughly the graphs of $f(x)=2.9^x$ and 2.9^{-x}.

Without using a calculator, find the following numbers.

18. $\log_{10}10^{-19}$ **19.** $\log_e e\sqrt[5]{e}$ **20.** $\log_2 16$

21. $\log_{10}\frac{10^3}{\sqrt{10}}$ **22.** $\ln\frac{e^2}{e^{21}}$ **23.** $\frac{\ln e^7}{\log_{11}121}$

24. $5^{\log_5 32.7}$ **25.** $e^{\ln\frac{9}{2}}$ **26.** $e^{\ln\sqrt[3]{27}}$

Using the rules of logarithms, rewrite the following expressions so that just one logarithm appears in each.

27. $3\log_2 x+\log_2 30+\log_2 y-\log_2 w$ **28.** $2\ln x-\ln y+a\ln w$

29. $12(\ln x+\ln y)$ **30.** $\log_3 e\times\ln 81+\log_3 5\times\log_5 w$

If you did not get most of those questions correct (in particular, questions 17 to 29), then you need to go back to the companion book, *Introduction to Exponents and Logarithms.*

2 Linearisation Using Logarithms

2.1 Introduction

Look at the graphs of three functions pictured below in Figure 1.

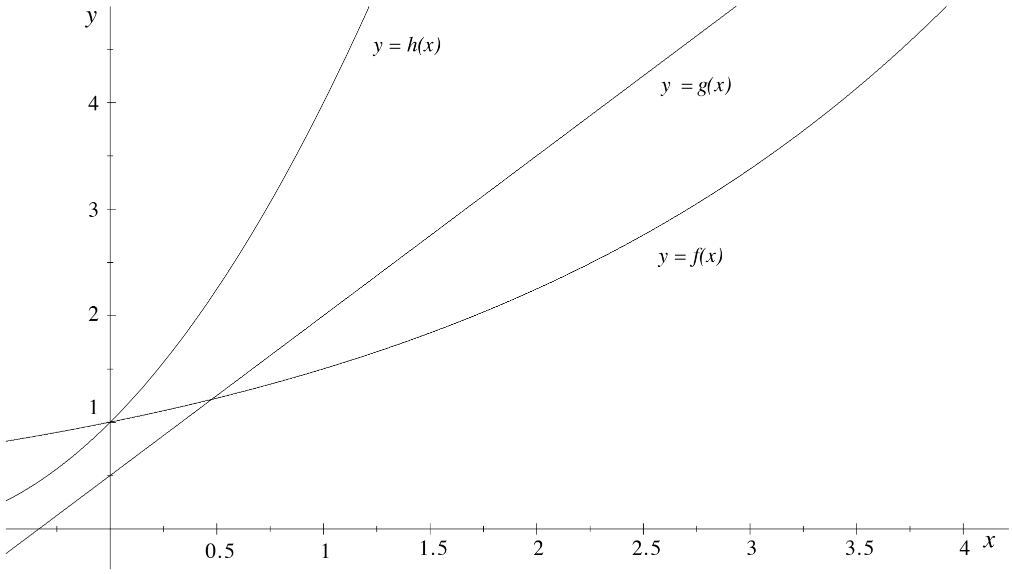

Figure 1: Graphs of three functions.

Do you think that it is possible, by looking at the graphs, to work out what the functions are? Well, for one of the functions it is relatively easy. The function $g(x)$ has as its graph a straight line, so it must be of the form $g(x) = mx + b$ where m is the slope of the line and b is the y-intercept, the point on the y-axis where the line cuts the y-axis. Even if we can't work out exactly what m and b are, we still know the general form of the function. In fact m is equal to $\frac{3}{2}$, and the y-intercept, b, is equal to $\frac{1}{2}$. Therefore the straight line is the graph of the function $g(x) = \frac{3}{2}x + \frac{1}{2}$.

On the other hand, we can't really see what the functions $f(x)$ and $h(x)$ are by simply inspecting the graph. We can't even really tell the general form of the functions. Actually $f(x) = e^{0.41x}$ and $h(x) = (x + 1)^2$, but it would be difficult to guess this by looking at the graphs.

The basis of this section is the observation that it is easy to recognise a straight line. If we see a graph which is a straight line then we know that it is the graph of a function of the form $y = mx + b$. If we see a graph that is curved then we know that it is not a straight line but, without more information, we cannot usually say much about what the form of the function is.

If we have a function of the form ae^{kx} (for example $y = 3.7e^{2x}$) or ax^b (for example $y = 3x^5$) then we can transform this function in a simple way to get a function of the form $f(x) = mx + b$, the graph of which is a straight line. We can tell from the position and slope of this straight line what the original function is.

2.2 Using a Log Transformation on Functions of the Form $y = ax^b$.

Consider a function of the form $y = ax^b$. Let's be specific and take the function to be $y = 2x^3$.

Take the logarithm (to any base, but we will use base e) of both sides of this equation, and we obtain the equation

$$\begin{aligned} \ln y &= \ln(2x^3) \\ &= \ln 2 + \ln(x^3) \\ &= \ln 2 + 3\ln x \end{aligned}$$

Now, what happens if in this equation, instead of writing $\ln y$ we write Y, and instead of writing $\ln x$ we write X? Then the equation $\ln y = \ln 2 + 3\ln x$ becomes $Y = \ln 2 + 3X$. As $\ln a$ is a constant, this is the equation of a straight line, with slope 3 and Y-intercept $\ln 2$.

We can best see what is going on here by making a table of values of x, y, X, and Y (Figure 2) and plotting some points on a graph (Figure 3).

x	1	2	3	4	5
y	2	16	54	128	250
$X = \ln x$	0	0.693	1.099	1.386	1.609
$Y = \ln y$	0.693	2.773	3.989	4.852	5.521

Figure 2: Table of data for function $y = 2x^3$ together with the values of $Y = \ln y$ and $X = \ln x$.

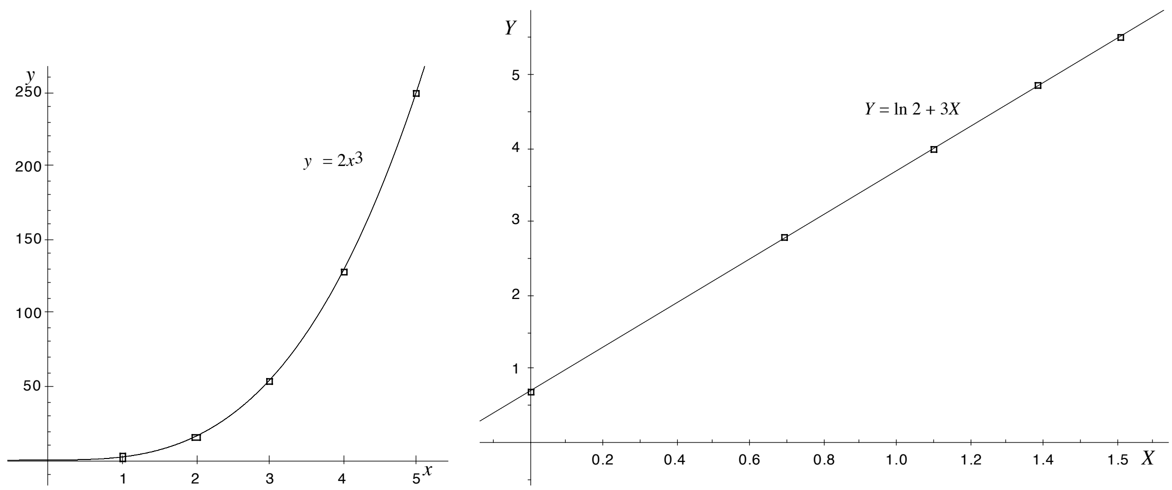

Figure 3: Graphs of $y = 2x^3$ and $Y = \ln 2 + 3X$.

By plotting our data as $\ln y$ against $\ln x$ we have come up with a graph which is a straight line, and therefore much easier to understand.

What usually happens in practise is this: Some data is obtained, usually through experiment or a sampling procedure. The person analysing the data thinks that they behave according to a function of the form $y = ax^b$, but is not sure of this, or of what values the constants a and b take. The researcher takes the data, calculates $Y = \ln y$ and $X = \ln x$ and plots them. If these data lie roughly along a straight line then the researcher knows that the data are behaving roughly according to the relationship $Y = \ln a + bX$, and by measuring the slope and intercept of a line of best fit is able to make an estimate of the values of $\ln a$ and b. a is found using the equation $a = e^{\ln a}$.

Example: The data in the table in Figure 4 are measurements made by a researcher studying blood sugar levels. The researcher suspects that the quantities x and y should

x	2.5	4.1	5.8	7.4	11.6	16.9	24.8	32.1	38.5
y	186	300	617	939	1294	3120	3890	5570	9370

Figure 4: Blood sugar levels research data.

behave according to a relationship of the form $y = ax^b$. Calculate $Y = \ln y$ and $X = \ln x$ and plot the data. Does this relationship hold and, if so, what are the the values of the constants a and b?

Solution: A table with the values of $Y = \ln y$ and $X = \ln x$ is given in Figure 5.

x	2.5	4.1	5.8	7.4	11.6	16.9	24.8	32.1	38.5
$X = \ln x$	0.92	1.41	1.76	2.00	2.45	2.82	3.21	3.47	3.65
y	186	300	617	939	1294	3120	3890	5570	9370
$Y = \ln y$	5.23	5.70	6.42	6.84	7.17	8.05	8.27	8.63	9.15

Figure 5: Blood sugar levels research data including transformed values.

A plot of the data, along with a line of best fit is shown in Figure 6.

It is clear that the data do fall (roughly) along a straight line. A 'line of best fit' has been drawn through the data. You may disagree with where the line has been drawn; its position is largely a subjective choice though in some contexts there are formulae which may be applied to tell us precisely where the line should be drawn. Anyway, let us suppose that we have agreed on the position of the line as in Figure 6. Since the data do fall along a straight line, we know that they are related by the formulae $y = ax^b$.

How do we find the constants a and b? Remember we are plotting X $(= \ln x)$ against Y $(= \ln y)$ as we took logarithms of both sides of the equation $y = ax^b$ to get $Y = \ln a + bX$ as shown below.

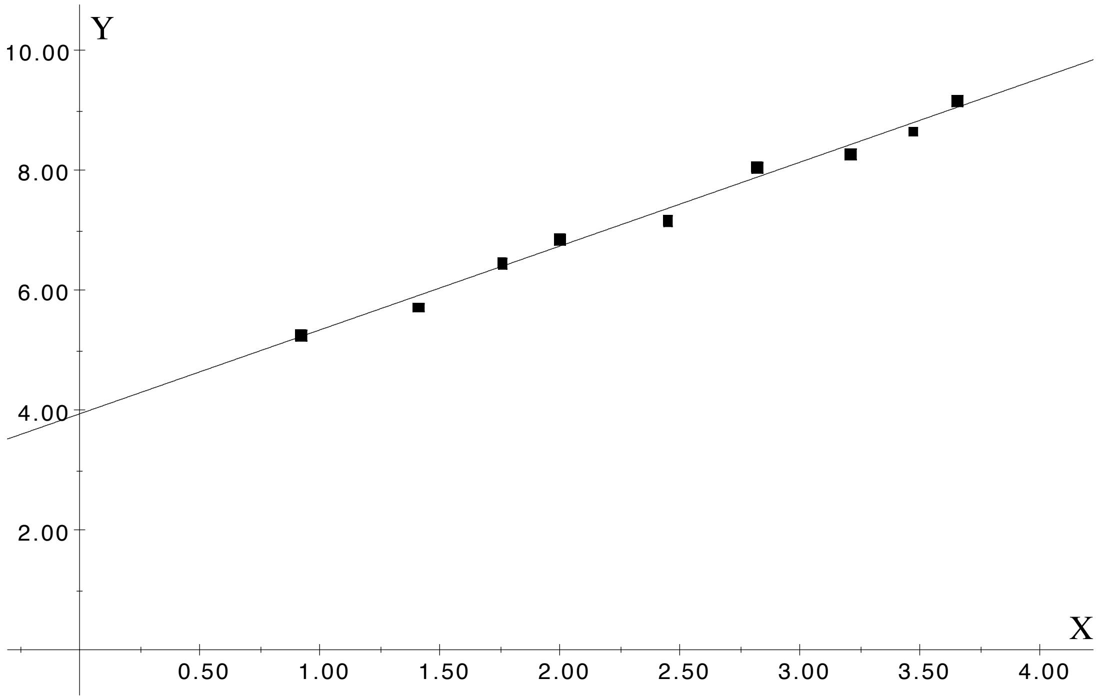

Figure 6: Plot of $Y = \ln y$ against $X = \ln x$.

$$\begin{aligned} y &= ax^b \\ \ln y &= \ln(ax^b) \qquad \text{taking logs of both sides} \\ &= \ln a + \ln(x^b) \\ &= \ln a + b\ln x \\ \text{So,} \quad Y &= \ln a + bX \end{aligned}$$

To find the slope we take two points *on the line* and use them to calculate the slope. It is very important to use two points actually on the line of best fit, not (X, Y) points (unless the line of best fit happens to pass through (X, Y) points). Since, in this case, the points (0.92, 5.23) and (1.76, 6.42) lie on the line, the slope of the line is therefore

$$\begin{aligned} b &\approx \frac{6.42 - 5.23}{1.76 - 0.92} \\ &\approx \frac{1.19}{0.84} \\ &\approx 1.4. \end{aligned}$$

Therefore $b \approx 1.4$. The best way to calculate the constant a is to substitute the coordinates of one of the points on the line into the equation $Y = \ln a + 1.4X$. If we use the point (1.76,6.42) then we obtain $6.42 \approx \ln a + 1.4(1.76)$, or $\ln a \approx 3.96$. Now, since $a = e^{\ln a}$, we get that $a \approx 52.46$.

Thus the data conforms approximately to the relationship $y = 52.5x^{1.4}$.

Exercises

1. The following data conform approximately to a relationship of the form $y = ax^b$.

Calculate X $(=\ln x)$ and Y $(=\ln y)$ and graph as in the example above. Determine the approximate values of the constants a and b.

x	1.63	3.13	3.86	6.05	9.68	19.3	35.2	54.6	89.1
y	22.3	35.3	52.6	71.0	132	231	438	798	1170

2.3 The Use of a Log Transformation on Functions of the Form $y = ae^{kx}$.

In the previous subsection we saw how we could use a log transformation to analyse data that conformed approximately to the relationship $y = ax^b$. In the same way we can use this transformation to analyse data conforming to the relationship $y = ae^{kx}$. Note carefully the difference between these two cases. In the former case the independent variable (in this example x) appears as the base of the exponential expression, whereas now the independent variable appears as part of the exponent.

Taking logarithms of both sides of the equation $y = ae^{kx}$ gives

$$\ln y = \ln a + kx.$$

We end up with a logarithm of the dependent variable, y, but not of the independent variable, x. Calling $Y = \ln y$ (but notice that we do not have to make any change to x), if we were to graph Y against x we would come up with a straight line. This line has slope k and Y-intercept $\ln a$.

Example: A scientist doing research into binary stars observes the data shown in the table in Figure 7.

The scientist suspects that the data conform to a relationship of the form $y = ae^{kx}$.

x	120	220	400	500	520	640	710	770	860
y	23	47	112	88	231	318	580	810	940

Figure 7: Binary star data.

Calculate Y $(=\ln y)$, and plot Y against x on a graph. Is the scientist's suspicion justified, and if so what are the values of the constants a and k?

Solution: A table including the transformed values Y is given in Figure 8.

x	120	220	400	500	520	640	710	770	860
y	23	47	112	88	231	318	580	810	940
$Y = \ln y$	3.14	3.85	4.72	4.48	5.44	5.76	6.36	6.70	6.85

Figure 8: Binary star data including values of $Y = \ln y$.

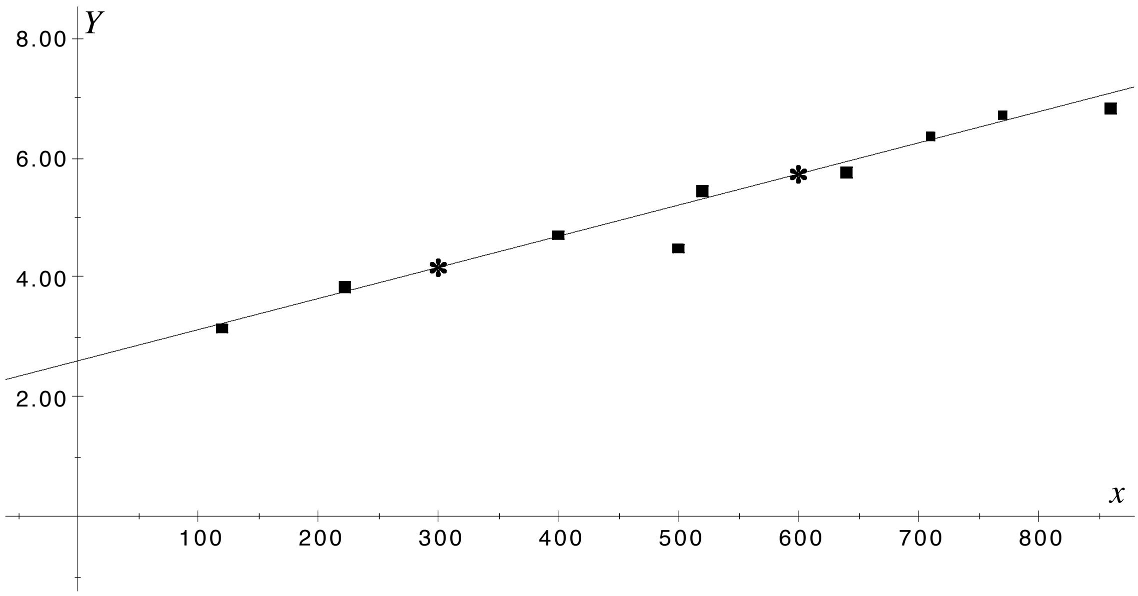

Figure 9: Plot of $Y = \ln y$ against x.

Figure 9 shows a plot of Y against x.

It is clear from this figure that, with one exception, the points lie approximately along a straight line, and so the scientist's suspicion is confirmed. The data do conform approximately to the relationship $y = ae^{kx}$. The exceptional point seems to go so far against the trend of the rest of the data that we have decided to ignore it.

We have drawn a line of best fit, and marked with asterixes two points on this line. They have coordinates (300, 4.2) and (600, 5.7). Note that these points are actually on the line, and not points (x, Y).

Since we have plotted $\ln y$ against x then the slope of the line is k. Using the chosen points we have

$$\begin{aligned} k &\approx \frac{5.7 - 4.2}{600 - 300} \\ &\approx 0.005. \end{aligned}$$

The best way to work out the value of a is to substitute either of the points (300,4.2) or (600,5.7) in the equation $Y = \ln a + x(0.005)$. Using the point (600,5.7) we get

$$\begin{aligned} 5.7 &= \ln a + (600)(0.005) \\ \ln a &= 2.7 \\ a &\approx 14.9 \end{aligned}$$

Therefore the data conform approximately to the relationship $y = 14.9e^{0.005x}$.

Exercise

2. The data in the table below are approximately related by the equation $y = ae^{kx}$. Calcluate $Y = \ln y$ and plot (x, Y). Find the approximate values of a and k.

x	0.58	2.10	3.14	4.08	4.52	5.96	7.46	8.72
y	0.34	0.53	0.96	1.45	1.58	3.52	5.92	9.95

2.4 Summary

Case 1

Data that conform (approximately) to a relationship of the form $y = ax^b$ will yield (approximately) a straight line when $Y = \ln y$ is plotted against $X = \ln x$. If two points (X_1, Y_1) and (X_2, Y_2) are chosen on the line of best fit then the constant b can be found from

$$b = \frac{Y_2 - Y_1}{X_2 - X_1}.$$

The constant $\ln a$ can be found by using this value of b and either one of the points (X_1, Y_1) or (X_2, Y_2) in the equation $Y = \ln a + bX$. Then a can be found using the equation $a = e^{\ln a}$.

Case2

Data that conform (approximately) to a relationship of the form $y = ae^{kx}$ will yield (approximately) a straight line when $y = \ln y$ is plotted against x. If two points (x_1, Y_1) and (x_2, Y_2) are chosen on the line of best fit then the constant k can be found using the equation

$$k = \frac{Y_2 - Y_1}{x_2 - x_1}.$$

The constant $\ln a$ can be found by using the value of k and either one of the points (x_1, Y_1) or (x_2, Y_2) in the equation $Y = \ln a + kx$. Again use $a = e^{\ln a}$ to find a.

2.5 Exercises

Each of the following sets of data conforms approximately to one of the relationships $y = ax^b$ or $y = ae^{kx}$. Find out to which of the relationships each data set conforms, and find the constants a and b or k.

3.

x	1.75	2.8	3.9	6.3	8.76	15.2	25.53	42.1
y	1.93	3.32	5.7	8.67	14.3	24.05	42.95	84.8

4.

x	1.12	2.22	3.08	3.94	5.0	6.04	7.14	8.18
y	18.2	35.3	66.9	103.8	209.1	388.6	694.1	1398

5.

x	1.00	1.98	3.28	4.58	6.0	6.9	7.82	9.0
y	7370	4045	2553	1218	655	466	272	179

6.

x	0.286	0.374	0.575	0.825	1.23	2.09
y	16.5	32.5	69.6	161.2	318.2	1099

3 Exponents, Logarithms and Calculus

3.1 Introduction

This section is an introduction to that part of calculus which involves exponential and logarithmic functions. You should only read it if you have some knowledge of calculus. Do not attempt this section if you have not seen any calculus before. The Mathematics Learning Centre booklet: *Introduction to Differential Calculus* may be useful if you need to learn calculus.

3.2 Derivatives of Logarithmic and Exponential Functions

We are going to apply the following results:

$$\frac{de^x}{dx} = e^x \tag{1}$$

$$\frac{d\ln x}{dx} = \frac{1}{x} \tag{2}$$

Remember $\ln x = \log_e x$.

We will use these results to differentiate functions which involve exponentials or logarithms. We will not derive these results here but if you want to see how they are derived then have a look at any calculus textbook.

We will make extensive use of the chain rule or composite function rule for differentiation and will give it here to remind you. Of course, we will use the other rules of differentiation as required.

Chain rule for differentiation

Have a look at the function $h(x) = (x^2 + 1)^{17}$. We can think of this function as being the result of combining two functions. If $g(x) = x^2 + 1$ and $f(u) = u^{17}$ then the result of substituting $g(x)$ into the function f is

$$h(x) = f(g(x)) = (g(x))^{17} = (x^2 + 1)^{17}.$$

Another way of representing this would be with a diagram like

$$x \overset{g}{\longmapsto} x^2 + 1 \overset{f}{\longmapsto} (x^2 + 1)^{17}.$$

We start off with x. The function g takes x to $x^2 + 1$, and the function f then takes $x^2 + 1$ to $(x^2 + 1)^{17}$. Combining two (or more) functions like this is called *composing* the functions, and the resulting function is called a *composite function.* For a more detailed discussion of composite functions you might wish to refer to the Mathematics Learning Centre booklet *Functions.*

Suppose that h is the composite of the differentiable functions $y = f(u)$ and $u = g(x)$. Then h is a differentiable function of x whose derivative is

$$h'(x) = f'(g(x))g'(x).$$

Another formulation of the chain rule, which gives us less information but may be easier to remember, is

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}.$$

Example: Differentiate e^{5x}.

Solution: This function is a composite function so we must use the composite function rule for differentiation together with result (1).

$h(x) = f(g(x))$ where $f(u) = e^u$ and $u = g(x) = 5x$. Since

$$f'(u) = e^u \qquad \text{and} \qquad g'(x) = 5$$

we have

$$h'(x) = f'(g(x))g'(x) = e^{5x}5 = 5e^{5x}.$$

Example: Find the derivative of $\ln(x^2 + 1)$.

Solution: Again we use the chain rule and result (2).

$h(x) = f(g(x))$ where $f(u) = \ln u$ and $u = g(x) = x^2 + 1$. Since

$$f'(u) = \frac{1}{u} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$h'(x) = f'(g(x))g'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

Example: Find $\dfrac{d(e^{3x^2})}{dx}$.

Solution: This is an application of the chain rule together with result (1).

$h(x) = f(g(x))$ where $f(u) = e^u$ and $u = g(x) = 3x^2$. Since

$$f'(u) = e^u \qquad \text{and} \qquad g'(x) = 6x$$

we have

$$h'(x) = f'(g(x))g'(x) = e^{3x^2} \cdot 6x = 6xe^{3x^2}.$$

Example: Differentiate $\ln(\cos x)$.

Solution: We solve this by using the chain rule and our knowledge of the derivative of $\ln x$.

$h(x) = f(g(x))$ where $f(u) = \ln u$ and $u = g(x) = \cos x$. Since

$$f'(u) = \frac{1}{u} \qquad \text{and} \qquad g'(x) = -\sin x$$

we have

$$h'(x) = f'(g(x))g'(x) = \frac{1}{\cos x} \cdot -\sin x = -\frac{\sin x}{\cos x} = -\tan x.$$

There are two shortcuts to differentiating functions involving exponents and logarithms. The examples in this section suggest the general rules

$$\frac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)} \tag{3}$$

$$\frac{d(\ln f(x))}{dx} = \frac{1}{f(x)} \cdot f'(x). \tag{4}$$

These rules arise from the chain rule and results (1) and (2), and can speed up the process of differentiation. It is not necessary that you remember them. If you forget, just use the chain rule as in the examples above.

Exercises
Differentiate the following functions.

1. $f(x) = \ln 2x^3$ **2.** $f(x) = e^{x^7}$ **3.** $f(x) = \ln(11x^7)$
4. $f(x) = e^{x^2+x^3}$ **5.** $f(x) = e^{7x^{-2}}$ **6.** $f(x) = \ln(e^x + x^3)$
7. $f(x) = e^{\frac{2x}{x^2+1}}$ **8.** $f(x) = \ln(\frac{5x-2}{x^2+3})$ **9.** $f(x) = \ln(e^x x^8)$

3.3 Logarithms and Exponents in Integration

Integration is the procedure which reverses differentiation. If $f(x)$ is a function, then the indefinite integral of f, written

$$\int f(x)\, dx,$$

is a function which, when differentiated, gives f.

In this section we are going to apply the following standard integrals:

$$\int e^x\, dx = e^x + C \tag{5}$$

$$\int \frac{1}{x}\, dx = \ln|x| + C \tag{6}$$

where C is an arbitrary constant. Results (5) and (6) are consequences of results (1) and (2) of section 2.1. We will also use the rules of integration, and, in particular, the chain rule.

Recall that if $h(x) = f(g(x))$ is the composite of $y = f(u)$ and $u = g(x)$. Then

$$h'(x) = f'(g(x))g'(x).$$

For example, if

$$h(x) = \ln(x^2 + 1) \quad \text{then} \quad h'(x) = \frac{1}{x^2+1}2x.$$

Now, if we think of integration as the reverse of diffentiation we get,

$$\int \frac{1}{x^2+1}2x\ dx = \ln(x^2 + 1) + C.$$

In general, if we integrate both sides of the equation $h'(x) = f'(g(x))g'(x)$ with respect to x we get,

$$\int f'(g(x))g'(x)\ dx = \int h'(x)\ dx = h(x) + C = f(g(x)) + C.$$

To use this result, you will need to be able to recognise when a function has this form. That is, the function to be integrated has the form of a product of two functions: One is a composite function, and the other is the derivative of the 'inner' function of the composite. Note that in some cases, this derivative is a constant.

We can best illustrate this method with some examples.

Example: Find $\int 3e^{3x}\ dx$.

Solution: In this example, e^{3x} is a composite function and the derivative of the 'inner' function is 3. So,

$$\int 3e^{3x}\ dx = \int e^{3x}3\ dx = \int f'(g(x))g'(x)\ dx \quad \text{with} \quad u = g(x) = 3x \quad \text{and} \quad f'(u) = e^u.$$

Since if $f'(u) = e^u$ then $f(u) = e^u$ we get,

$$\int 3e^{3x}\ dx = f(g(x)) = e^{3x} + C.$$

If you have doubts about this, then differentiate the answer to check it out.

Example: Find $\int 4xe^{2x^2}\ dx$

Solution: In this example, the composite function is e^{2x^2}, and the derivative of the 'inner' function is $4x$. So,

$$\int 4xe^{2x^2} = \int e^{2x^2}4x\ dx = \int f'(g(x))g'(x)\ dx \quad \text{with} \quad u = g(x) = 2x^2 \quad \text{and} \quad f'(u) = e^u.$$

Since if $f'(u) = e^u$ then $f(u) = e^u$ we get,

$$\int 4xe^{2x^2}\ dx = \int e^{2x^2}4x\ dx = e^{2x^2} + C.$$

Again, check the result by differentiating.

Example: Find $\int (\ln x)^2 \cdot \frac{1}{x}\, dx$.

Solution: Here the composite function is $(\ln x)^2$ and the derivative of the 'inner' function is $\frac{1}{x}$. So,

$$\int (\ln x)^2 \cdot \frac{1}{x}\, dx = \int f'(g(x))g'(x)\, dx \quad \text{with} \quad u = g(x) = \ln x \quad \text{and} \quad f'(u) = u^2.$$

Since if $f'(u) = u^2$ then $f(u) = \frac{u^3}{3}$ we get,

$$\int (\ln x)^2 \cdot \frac{1}{x}\, dx = f(\ln x) + C = \frac{(\ln x)^3}{3} + C.$$

Example: Find $\int \frac{2x-4}{x^2-4x+1}\, dx$.

Solution: Here the composite function is $\frac{1}{x^2-4x+1}$ and the derivative of the 'inner' function is $2x-4$. So,

$$\int \frac{2x-4}{x^2-4x+1}\, dx = \int f'(g(x))g'(x)\, dx \quad \text{with} \quad u = g(x) = x^2-4x+1 \quad \text{and} \quad f'(u) = \frac{1}{u}.$$

Since if $f'(u) = \frac{1}{u}$ then $f(u) = \ln|u|$ we get,

$$\int \frac{2x-4}{x^2-4x+1}\, dx = f(x^2-4x+1) + C = \ln|x^2-4x+1| + C.$$

The examples above suggest the following rules, which can be very useful when integrating certain functions.

$$\int f'(x)e^{f(x)}\, dx = e^{f(x)} + C \tag{7}$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C. \tag{8}$$

These rules actually follow from results (3) and (4). It is not necessary that you remember them. If you forget, just use a substitution as in the examples above.

Example: Find $\int x^5 e^{x^6}\, dx$.

Solution: This one is a little tricky as the composite function is e^{x^6} but the derivative of the 'inner' function is $6x^5$ and we have only x^5 in the product. We can get around this difficulty by writing

$$\int x^5 e^{x^6}\, dx = \frac{1}{6}\int 6x^5 e^{x^6}\, dx.$$

Now we can complete the solution as before

$$\begin{aligned}\int x^5 e^{x^6}\,dx &= \frac{1}{6}\int 6x^5 e^{x^6}\,dx \\ &= \frac{1}{6}e^{x^6} + C.\end{aligned}$$

Example: Find $\displaystyle\int \frac{4x+2}{x^2+x}\,dx$.

Solution: Again we need to make a minor change to the integrand first as the derivative of the 'inner' function is $2x+1$.

$$\begin{aligned}\int \frac{4x+2}{x^2+x}\,dx &= 2\int \frac{2x+1}{x^2+x}\,dx \\ &= 2\ln|x^2+x| + C.\end{aligned}$$

Exercises
Evaluate the following indefinite integrals.

10. $\int \frac{2}{2x+1}\,dx$

11. $\int 2xe^{x^2+2}\,dx$

12. $\int 3x^2 e^{x^3}\,dx$

13. $\int \frac{2x+2}{x^2+2x+7}\,dx$

14. $\int \frac{x+2}{x^2+4x+1}\,dx$

15. $\int e^{-2x}\,dx$

16. $\int \frac{1}{x^3}e^{-\frac{1}{x^2}}\,dx$

17. $\int \frac{1}{1-x}\,dx$

18. $\int \cos x e^{\sin x}\,dx$

3.4 Logarithmic Differentiation

Logarithmic differentiation is a useful technique which can simplify the process of finding the derivatives of certain functions. Typically it is useful when the function to be differentiated involves products or quotients of a (possibly) large number of (possibly) complicated factors or for exponential functions like $f(x) = 2^x$. The best way of illustrating the method is by example.

Example: Find the derivative of $f(x) = x^3 \sin x(2x+1)^{-4}$.

Solution: We take logarithms (to base e) of both sides.

$$\begin{aligned}f(x) &= x^3(2x+1)^{-4}\sin x \\ \ln f(x) &= \ln(x^3(2x+1)^{-4}\sin x) \\ &= \ln(x^3) + \ln((2x+1)^{-4}) + \ln(\sin x) \\ &= 3\ln x - 4\ln(2x+1) + \ln(\sin x)\end{aligned}$$

We now differentiate both sides of this equation with respect to x. To differentiate $\ln f(x)$ we make use of (4):

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

Differentiating the right hand side we get

$$\frac{f'(x)}{f(x)} = \frac{3}{x} - \frac{4(2)}{(2x+1)} + \frac{\cos x}{\sin x}.$$

Multiplying both sides of this last equation by $f(x)$ completes the solution.

$$\begin{aligned}
f'(x) &= f(x)\left(\frac{3}{x} - \frac{8}{(2x+1)} + \frac{\cos x}{\sin x}\right) \\
&= x^3(2x+1)^{-4}\sin x\left(\frac{3}{x} - \frac{8}{(2x+1)} + \frac{\cos x}{\sin x}\right) \\
&= 3x^2(2x+1)^{-4}\sin x + x^3(2x+1)^{-4}\cos x - 8x^3(2x+1)^{-5}\sin x \\
&= \frac{3x^2\sin x}{(2x+1)^4} + \frac{x^3\cos x}{(2x+1)^4} - \frac{8x^3\sin x}{(2x+1)^5}
\end{aligned}$$

We can also use logarithmic differentiation to differentiate functions of the form $f(x) = 2^x$.

$$\begin{aligned}
f(x) &= 2^x \\
\ln f(x) &= \ln(2^x) \\
&= x\ln 2 \\
\frac{1}{f(x)}f'(x) &= \ln 2 \\
\text{So,}\quad f'(x) &= f(x)\ln 2 \\
&= (\ln 2)2^x
\end{aligned}$$

Exercises

Use logarithmic differentiation to find the derivatives of the following functions.

19. $f(x) = \frac{x^7\cos x}{(x+1)^2}$ **20.** $f(x) = 3^{-x}$ **21.** $f(x) = x^{-3}(x^2+3x+1)^4 5^{-x}$

3.5 Summary

The following basic results hold.

$$\begin{aligned}
\frac{de^x}{dx} &= e^x \\
\frac{d\ln x}{dx} &= \frac{1}{x} \\
\int e^x\, dx &= e^x + C \\
\int \frac{1}{x}\, dx &= \ln|x| + C
\end{aligned}$$

The following formulae are useful when differentiating or integrating.

$$\begin{aligned}
\frac{de^{f(x)}}{dx} &= f'(x)e^{f(x)} \\
\frac{d(\ln f(x))}{dx} &= \frac{f'(x)}{f(x)} \\
\int f'(x)e^{f(x)}\,dx &= e^{f(x)} + C \\
\int \frac{f'(x)}{f(x)}\,dx &= \ln|f(x)| + C.
\end{aligned}$$

Logarithmic differentiation is a useful technique for differentiating expressions involving a large number of factors.

3.6 Exercises

Calculate the derivatives of the following functions.

22. $f(x) = \ln 3x^4$ **23.** $f(x) = e^{2x^9}$ **24.** $f(x) = \ln 3x^6$

25. $f(x) = e^{x^5 - x^2}$ **26.** $f(x) = \ln 4x^{-6}$ **27.** $f(x) = \ln(e^{x^2} + 2x^4)$

Use logarithmic differentiation to find the derivatives of the following functions.

28. $f(x) = \dfrac{x^4(x+x^2)^3}{(x-1)^2}$ **29.** $f(x) = 4^x e^{x^2}$ **30.** $f(x) = x^5(x^5 + 3x^2 + 1)^3 2^{-x}$

Evaluate the following indefinite integrals.

31. $\displaystyle\int 4x^3 e^{x^4}\,dx$ **32.** $\displaystyle\int \frac{3x^2+7}{x^3+7x+7}\,dx$ **33.** $\displaystyle\frac{1}{2}\int x^3 e^{x^4+1}\,dx$

34. $\displaystyle\int \frac{\sin x}{\cos x + 5}\,dx$ **35.** $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$ **36.** $\displaystyle\int \frac{1}{x^4} e^{-\frac{1}{x^3}}\,dx$

37. $\displaystyle\int e^x e^{e^x}\,dx$ **38.** $\displaystyle\int \frac{5}{x}\,dx$ **39.** $\displaystyle\int x e^{-x^2}\,dx$

4 Exponential Growth and Decay

4.1 Introduction

To introduce the topic of exponential growth and decay we can consider the growth in the population of a single-celled organism. Suppose that the animal reproduces by each single cell splitting into two cells, and that this occurs regularly, say every hour. Let's imagine starting off with just one of these organisms and looking at how the population grows as time goes by (see Figure 10). If none die then after one hour the organism will

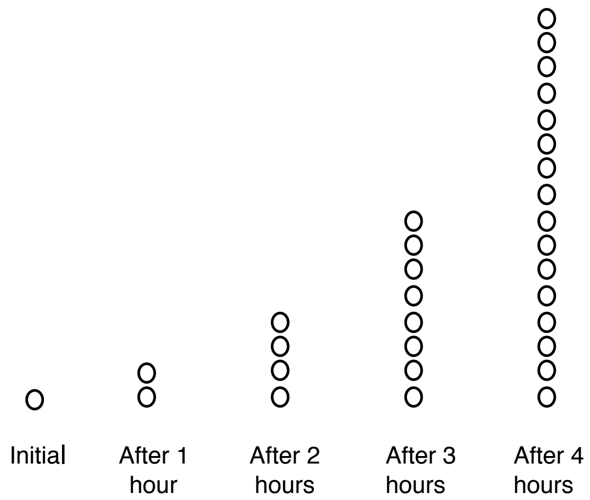

Figure 10: Growth of single celled organisms.

have reproduced and there will be two of them. After a further hour has elapsed each of these two animals will have itself reproduced and there will now be $4 = 2^2$ of them. After another hour there will be $8 = 2^3$ of the animals, and so on. This type of increase in the population is called *exponential growth*.

Now let us try to write down what is going on here in symbols. Instead of assuming that the animals reproduce every hour, suppose they reproduce every T units of time. The unit of time could be seconds, minutes, hours or even years, we don't care as long as we stick to the same measuring units throughout the problem.

Let's measure time from some fixed moment, which we will call $t = 0$, and we will suppose now that instead of beginning with one organism we started off with an initial population of P_0. This means that at time $t = 0$, the population of the animals is P_0.

What is the population after one period of time T?

Well, at time T each of the P_0 single cells has split into two cells, so the population is $2 \times P_0$. We can represent this by the equation $P(T) = P_0 \times 2$.

After a further period of time T, when $t = 2T$, each of the $2 \times P_0$ cells has split and the population is now $4 \times P_0 = P_0 \times 2^2$. Thus $P(2T) = P_0 \times 2^2$.

Proceeding like this we see that the population at time nT is $P_0 \times 2^n$. The equation which represents the growth of the cell culture is thus $P(nT) = P_0 \times 2^n$. If we make the substitution $t = nT$, then we obtain the equation

$$P(t) = P_0 \times 2^{\frac{t}{T}}.$$

Since $2 = e^{\ln 2}$, this equation can be rewritten as

$$\begin{aligned} P(t) &= P_0 \times (e^{\ln 2})^{\frac{T}{t}} \\ &= P_0 \times e^{\frac{\ln 2}{T} \times t} \\ &= P_0 e^{kt}, \qquad k = \frac{\ln 2}{T}. \end{aligned}$$

The equation

$$P(t) = P_0 e^{kt}$$

(with $k > 0$) is the equation for exponential growth.

Let's look at a second example, a certain radioactive isotope, isotope X, the atoms of which decay into another isotope, isotope Y. Suppose that in an interval of time T, half of the atoms of isotope X decay into atoms of isotope Y. If the initial amount of isotope X was P_0 then after time T the amount of isotope Y left is $P_0 \times \frac{1}{2}$.

In a similar way to the previous example we can represent this by $P(T) = P_0 \times \frac{1}{2}$ or $P(T) = P_0 \times 2^{-1}$.

After a further interval of time T (so that $t = 2T$), half of the remaining atoms of isotope X have decayed, and the amount of isotope X remaining is $P_0 \times \frac{1}{2} \times \frac{1}{2}$. This can be represented by the equation $P(2T) = P_0 \times 2^{-2}$.

Proceeding in this way we find that after a time interval nT the amount of isotope X remaining is equal to $P(nT) = P_0 \times 2^{-n}$. Have a look at Figure 11.

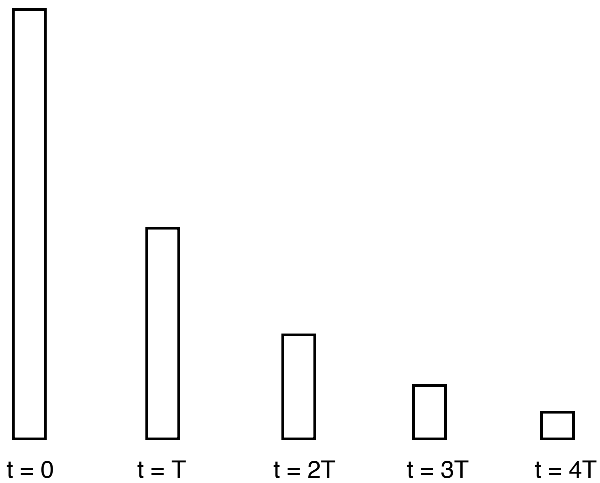

Figure 11: Relative amounts of isotope X remaining for various time periods.

If we make the substitution $t = nT$, then we obtain the equation

$$P(t) = P_0 \times 2^{\frac{-t}{T}}.$$

Since $2 = e^{\ln 2}$, this equation can be rewritten as

$$\begin{aligned} P(t) &= P_0 \times e^{-\frac{\ln 2}{T} \times t} \\ &= P_0 e^{kt}, \qquad k = -\tfrac{\ln 2}{T}. \end{aligned}$$

The equation

$$P(t) = P_0 e^{kt}$$

(with $k < 0$) is the equation for exponential decay.

You should notice that the equation representing exponential growth is the same as that representing exponential decay, except that for growth the constant k is greater than zero, and $k < 0$ for decay.

The speed of exponential decay is often specified by giving the *half life* of the quantity that is decaying. The half life is the time it takes for a quantity to reduce to half of its initial size. If we are told the half life of a decaying quantity then we are able to calculate the constant k in the equation $P(t) = P_0 e^{kt}$.

Example: If a quantity P is decaying exponentially with a half life of 250 years, find the equation expressing the size of this quantity at time t.

Solution: In 250 years time the size of the quantity will be half its present size. If P_0 is the initial size of the quantity then, when $t = 250$, $P = \frac{1}{2}P_0$. So,

$$\begin{aligned} P(250) = \frac{1}{2}P_0 &= P_0 e^{k \times 250} \\ \frac{1}{2} &= e^{k \times 250} \\ \ln \frac{1}{2} &= k \times 250 \\ k &= \frac{\ln \frac{1}{2}}{250}. \end{aligned}$$

We have found the value of the constant k, and so the equation representing the decay of the quantity is

$$P(t) = P_0 e^{\frac{\ln \frac{1}{2}}{250} t}.$$

Since $\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$, the equation representing the decay of the quantity can also be written as

$$P(t) = P_0 e^{\frac{-\ln 2}{250} t}.$$

Example: Under certain laboratory conditions the population of a particular single celled organism is known to grow exponentially. A researcher begins an experiment with the organism and observes that after 3 hours the population has increased to a level 1.5 times that of the initial population. How long after the beginning of the experiment will the population double?

Solution: We know that the population of the organism is given by

$$P(t) = P_0 e^{kt},$$

where P_0 is the initial population and t is the time which has elapsed since the beginning of the experiment. We also know that when $t = 3$ the population is $\frac{3}{2} \times P_0$, ie

$$P(3) = \frac{3}{2} \times P_0 = P_0 e^{k3},$$

so that $e^{3k} = \frac{3}{2}$. Taking natural logarithms of both sides of this equation gives

$$\begin{aligned} 3k &= \ln \frac{3}{2} \text{ or} \\ k &= (\frac{1}{3}) \ln \frac{3}{2}. \end{aligned}$$

We have worked out the value of the constant k, but don't reach for your calculator just yet. It is more convenient for us to leave k in this form for now. Substituting this value of k in the expression for exponential growth we obtain

$$P(t) = P_0 e^{\frac{1}{3}(\ln \frac{3}{2})t}.$$

We are after the value of t which makes $P(t) = 2P_0$, or

$$\begin{aligned} 2P_0 &= P_0 e^{\frac{1}{3}(\ln \frac{3}{2})t} \\ 2 &= e^{\frac{1}{3}(\ln \frac{3}{2})t} \\ \ln 2 &= \frac{1}{3}(\ln \frac{3}{2})t \\ t &= \frac{3 \ln 2}{\ln \frac{3}{2}} \end{aligned}$$

If necessary this number may be found in decimal form using a calculator (5.1285 to 4 decimal places). However in most circumstances it is preferable to leave it in the form given here.

Exercises

1. The size of a quantity P at time t is given by $P(t) = 1300e^{2.1t}$, where t is measured in seconds.
(a) Is P increasing or decreasing with time?
(b) What is the initial value of P (ie what is $P(0)$)?
(c) What is the value of P after 1.5 seconds?
(d) In how many seconds (measured from $t = 0$) will the quantity be equal to 2000?

2. A quantity W is known to decay exponentially. Initially W is equal to 22.5, and after 3 hours W has decreased to 15.5.
(a) Write down an expression for $W(t)$, the size of the quantity at time t.
(b) How long before the quantity has decreased to 10.0?

Example (Radioactive Carbon dating): Carbon-14 is a radioactive isotope of carbon which is produced by activity in the upper atmosphere. Carbon-14 decays in an exponential fashion to produce nitrogen. The half life of Carbon-14 is approximately 5730 years. Each living organism contains Carbon-14, which is ingested as part of the normal life cycle of the organism, and whilst the organism is alive the level of Carbon-14 in it remains roughly constant. When the organism dies no more Carbon-14 is ingested by it and the level decreases. Measurement of the level of Carbon-14 in ancient dead organic matter, and comparison of the level with that contained in similar living matter, yields an estimate of the age of the dead matter.

Analysis of some human remains found in the Gibson Desert shows that the level of Carbon-14 in the bones was 0.358 times that which would be found in the bones of living humans. What is the approximate age of these remains?

Solution: If C represents the level of Carbon-14 in the remains, and measuring time t from the moment of death of the human, then

$$C(t) = C_0 e^{kt}.$$

After 5730 years (the half life of Carbon-14) C is half its original level, so

$$\begin{aligned} C(5730) = \frac{1}{2}C_0 &= C_0 e^{k\times 5730} \\ \frac{1}{2} &= e^{k\times 5730} \\ \ln\frac{1}{2} &= k \times 5730 \\ k &= \frac{\ln\frac{1}{2}}{5730} \end{aligned}$$

and

$$C(t) = C_0 e^{\frac{\ln\frac{1}{2}}{5730}t}.$$

So far all we have done is use our information about the half life of Carbon-14 to find the formula by which Carbon-14 decays. Our solution so far would be the same for any problem involving radiocarbon dating. To find the age of these particular bones we need to use the fact that the level of Carbon-14 has been reduced to 0.358 of the initial level. So

$$\begin{aligned} 0.358C_0 &= C_0 e^{\frac{\ln\frac{1}{2}}{5730}t} \\ 0.358 &= e^{\frac{\ln\frac{1}{2}}{5730}t} \\ \ln 0.358 &= \frac{\ln\frac{1}{2}}{5730}t \\ t &= \frac{\ln 0.358}{\frac{\ln\frac{1}{2}}{5730}} \\ &\approx 8600. \end{aligned}$$

The remains are approximately 8600 years old.

Exercise

3. Plutonium-239 is one of the most toxic substances known. The safe level of Plutonium-239 that can be ingested by one human has been set at 0.64 micrograms, or 6.4×10^{-7} grams. It decays exponentially, and has a half life of 243,000 years. In January 1968 an atomic bomb was lost in Greenland, and it has been estimated that 400 grams of Plutonium-239 was released into the marine environment. Calculate the length of time which must pass before the 400 grams of Plutonium-239 has decayed to an amount which is considered safe for ingestion by one human being.

4.2 Calculus and Exponential Growth and Decay

This subsection is intended for people who have some knowledge of calculus, and should be omitted by anyone who has not.

If P is a quantity that is growing or decaying exponentially then the size of the quantity P at time t is described by the equation $P(t) = P_0e^{kt}$, where a negative constant k means decay and a positive k means growth. It is natural to ask about the rate of change in the size of the quantity.

Questions about rates of change in mathematics are usually answered by means of calculus. The rate of increase or decrease in the quantity P is equal to $\frac{dP}{dt}$.

If $P(t) = P_0e^{kt}$,then $\frac{dP}{dt} = kP_0e^{kt} = kP(t)$.

Thus if the size of the quantity at time t, $P(t)$, is given by the equation $P(t) = P_0e^{kt}$ then *the rate of growth (or decay) of the quantity is proportional to the size of the quantity*. In symbols,

$$\begin{aligned} \frac{dP}{dt} &\propto P \quad \text{or, introducing the proportionality constant } k, \\ \frac{dP}{dt} &= kP. \end{aligned}$$

On the other hand, suppose we know that a quantity P increases (or decreases) at a rate proportional to P (this can be expressed symbolically by the equation $\frac{dP}{dt} = kP$ or $P'(t) = kP$). Then

$$\begin{aligned} \frac{P'(t)}{P(t)} &= k \\ \int \frac{P'(t)}{P(t)}\, dt &= \int k\, dt \\ \ln P(t) &= kt + C \\ P(t) &= e^{kt+C} = e^C e^{kt}. \end{aligned}$$

When $t = 0$ this gives $P(0) = e^C e^0 = e^C$, and we arrive at the equation

$$P(t) = P(0)e^{kt} = P_0e^{kt}.$$

What we have shown here is that exponential growth or decay is *characterised* by the property that the rate of change of the quantity under consideration is proportional to the size of the quantity. If we are studying a quantity that we know increases (or decreases) at a rate which is proportional to the size of the quantity then this quantity is growing (or decaying) exponentially.

The examples with which we introduced exponential growth and decay may make more sense to you now. If a single celled animal reproduces every hour, and none of the animals in the culture die, then the rate of increase in the population is proportional to the size of the population—if there are twice as many organisms then they will have twice as many offspring. This is why the population of such (imaginary) organisms grows exponentially.

Example: A scientist is studying a cell culture and finds that the the population P of the culture increases at a rate given by the equation

$$\frac{dP}{dt} = 1.7P.$$

How long does it take for the population to increase to a level 2.5 times that of the initial population?

Solution: Although we know that the solution is

$$P(t) = P_0 e^{1.7t}$$

we will work it out as follows. Rearranging the equation $P'(t) = 1.7P$ gives

$$\frac{P'(t)}{P(t)} = 1.7.$$

Taking the indefinite integral (with respect to t) of both sides of this equation gives

$$\begin{aligned}
\int \frac{P'(t)}{P(t)} dt &= \int 1.7 dt \\
\ln P(t) &= 1.7t + C \\
e^{\ln P(t)} &= e^{1.7t+C} \\
P(t) &= e^C e^{1.7t}.
\end{aligned}$$

When $t = 0$ this equation reduces to $P(0) = e^C$, so e^C is equal to the initial population of the culture and we arrive at the equation

$$P(t) = P_0 e^{1.7t}.$$

We are asked to find how long until the population of the culture grows to a level 2.5 times that of the initial population. In other words, when is $P(t) = 2.5P_0$?

Well

$$\begin{aligned}
P(t) = 2.5P_0 &= P_0 e^{1.7t} \\
2.5 &= e^{1.7t} \\
\ln 2.5 &= 1.7t \\
t &= \frac{\ln 2.5}{1.7}.
\end{aligned}$$

The population reaches a level of 2.5 times that of the initial population at a time of $\frac{\ln 2.5}{1.7}$.

Exercise

4. (Attempt only if you know some calculus.) A quantity P being measured in an experiment increases according to the equation $\frac{dP}{dt} = 4.5P$. The initial amount of the quantity was 480 units. Find an expression for the amount $P(t)$ of the quantity at time t.

4.3 Summary

The exponential growth or exponential decay of a quantity P are both represented by the equation

$$P(t) = P_0 e^{kt}$$

where

$$\begin{aligned} P(t) &= \text{the size of the quantity P at time } t, \\ P_0 &= \text{the initial size of the quantity (at time } t = 0\text{)}, \\ k &= \text{a constant, } k > 0 \text{ for growth and } k < 0 \text{ for decay}, \\ t &= \text{time, measured in any convenient units.} \end{aligned}$$

The rate of exponential decay is often specified by stating the *half life* of the quantity. The half life is the time for taken for a quantity to decay to half its initial size.

If a quantity P is growing (or decaying) in an exponential fashion then the rate of growth (or decay) of the quantity is proportional to the size of the quantity. In symbols

$$\begin{aligned} \frac{dP}{dt} &\propto P \quad \text{or} \\ \frac{dP}{dt} &= kP. \end{aligned}$$

On the other hand, any quantity that grows (or decays) in such a fashion is growing (or decaying) exponentially.

In short, exponential growth (or decay) is *characterised* by the property that $\frac{dP}{dt} = kP$.

4.4 Exercises

5. A quantity A is known to grow exponentially. After 2 minutes of a laboratory experiment the quantity was measured to be 153, and exactly one minute later the quantity was measured to be 247.
(a) What was the initial value of A?
(b) Write down an expression for $A(t)$.
(c) What is the size of A after 7 minutes?
(d) When will the value of $A(t)$ be 2000?

6. A quantity B is known to grow exponentially. At the beginning of an experiment the quantity B was measured to be 47.9, and after 6.7 minutes B had increased to 102.
(a) Write down an expression for $B(t)$, the size of the quantity at time t.
(b) How long before the quantity has increased to 500?

7. Radium decays exponentially, with a half life of 1466 years. This means that it takes 1466 years for the amount of radium in a given sample to decrease to half its initial level.
(a) Find the equation governing the decay of radium.
(b) How long does it take for the amount of radium in a given sample to decrease to one fifth of its initial level?

5 Solutions to Exercises

5.1 Solutions to Exercises from Section 1

1. $9^{\frac{1}{2}} = \sqrt{9} = 3$

2. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$

3. $(\frac{1}{5})^{-2} = \frac{1}{(\frac{1}{5})^2} = 25$

4. $(3^{-1})^2 = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

5. $(\frac{5}{2})^{-2} = (\frac{2}{5})^2 = \frac{4}{25}$

6. $(-8)^{\frac{3}{2}}$ is not defined.

7. $(\frac{-27}{8})^{\frac{2}{3}} = ((\frac{-27}{8})^{\frac{1}{3}})^2 = (\frac{-3}{2})^2 = \frac{9}{4}$

8. $5^{27}5^{-24} = 5^{27-24} = 5^3 = 125$

9. $8^{\frac{1}{2}}2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$

10. $(-125)^{\frac{2}{3}} = ((-125)^{\frac{1}{3}})^2 = (-5)^2 = 25$

11. $\frac{3^{n+2}}{3^{n-2}} = 3^{n+2-(n-2)} = 3^4 = 81$

12. $\sqrt{\frac{16}{x^6}} = (\frac{16}{x^6})^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{x^{6\times\frac{1}{2}}} = \frac{4}{x^3}$

13. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2 + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2 = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$

14.

$$\begin{aligned}
(x^2+y^2)^{\frac{1}{2}} - x^2(x^2+y^2)^{-\frac{1}{2}} &= (x^2+y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{(x^2+y^2)^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}} - x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{x^2+y^2-x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{y^2}{(x^2+y^2)^{\frac{1}{2}}}
\end{aligned}$$

15. $\frac{x^{\frac{1}{2}}+x}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} = 1 + x^{\frac{1}{2}}$

16.

$$\begin{aligned}
(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}}) &= u^{\frac{1}{3}}u^{\frac{2}{3}} + u^{\frac{1}{3}}(uv)^{\frac{1}{3}} + u^{\frac{1}{3}}v^{\frac{2}{3}} - v^{\frac{1}{3}}u^{\frac{2}{3}} - v^{\frac{1}{3}}(uv)^{\frac{1}{3}} - v^{\frac{1}{3}}v^{\frac{2}{3}} \\
&= u - v
\end{aligned}$$

17. The graphs are drawn in Figures 12 and 13 below. Notice that the graph of $f(x) = 2.9^x$ is very close to the graph of $f(x) = 3^x$, and similarly for the other pair of graphs.

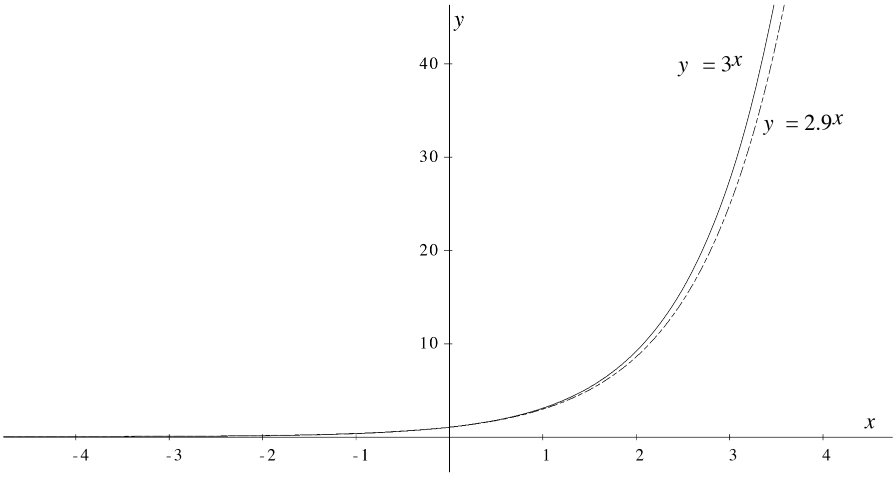

Figure 12: Graphs of $y = 3^x$ and $y = 2.9^x$.

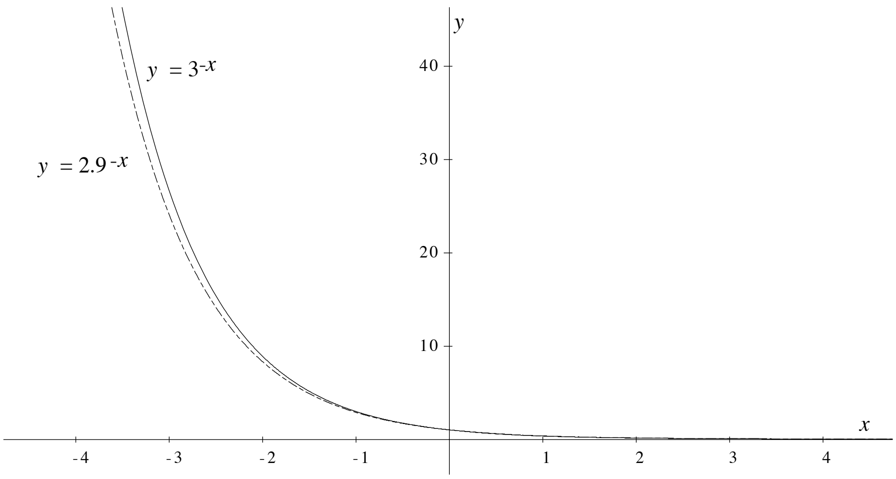

Figure 13: Graphs of $y = 3^{-x}$ and $y = 2.9^{-x}$.

18. $\log_{10} 10^{-19} = -19$

19. $\log_e e\sqrt[5]{e} = \log_e e^{\frac{6}{5}} = \frac{6}{5}$

20. $\log_2 16 = \log_2 2^4 = 4$

21. $\log_{10} \frac{10^3}{\sqrt{10}} = \log_{10} 10^{3-\frac{1}{2}} = \frac{5}{2}$

22. $\ln \frac{e^2}{e^{21}} = \ln e^{2-21} = -19$

23. $\frac{\ln e^7}{\log_{11} 121} = \frac{7}{\log_{11} 11^2} = \frac{7}{2}$

24. $5^{\log_5 32.7} = 32.7$

25. $e^{\ln \frac{9}{2}} = \frac{9}{2}$

26. $e^{\ln \sqrt[3]{27}} = \sqrt[3]{27} = 3$

27. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w = \log_2 \frac{30x^3y}{w}$

28. $2\ln x - \ln y + a\ln w = \ln x^2 - \ln y + \ln w^a = \ln \frac{x^2w^a}{y}$

29. $12(\ln x + \ln y) = \ln(xy)^{12}$

30. $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w = \log_3 81 + \log_3 w = 4 + \log_3 w$

5.2 Solutions to Exercises from Section 2

1. A table of the values of $Y = \ln y$ and $X = \ln x$ is given in Figure 14.

$X = \ln x$	0.49	1.14	1.35	1.80	2.27	2.96	3.56	4.00	4.49
$Y = \ln y$	3.10	3.56	3.96	4.26	4.88	5.44	6.08	6.68	7.06

Figure 14: Values of $X = \ln x$ and $Y = \ln y$ for exercise 2.1.

The data have been plotted and a line of best fit drawn in Figure 15. The points

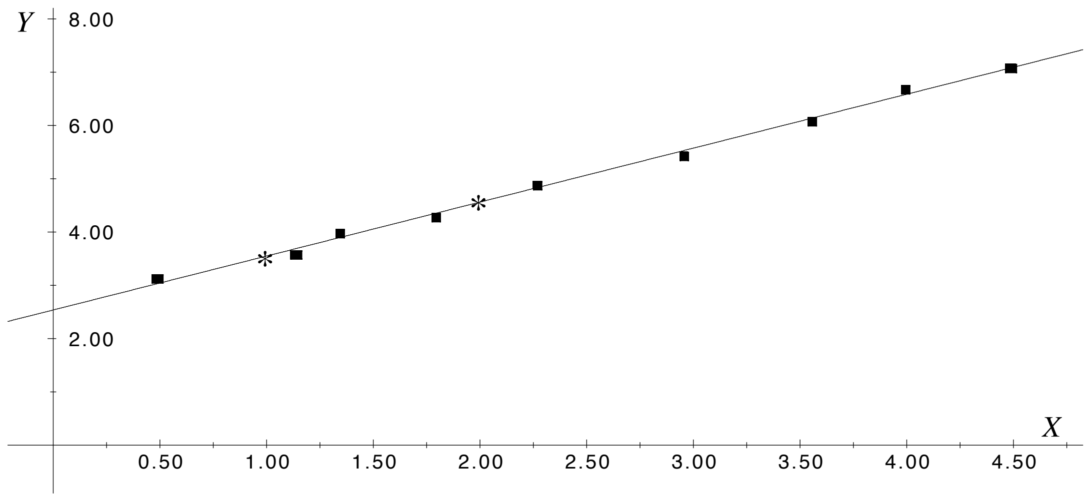

Figure 15: Plot of $Y = \ln y$ against X for exercise 2.1

$(1, 3.53)$ and $(2, 4.55)$, each marked with an asterix, are on the line of best fit. We calculate the slope of this line, b,

$$\begin{aligned} b &= \frac{4.55 - 3.53}{2 - 1} \\ &\approx \frac{1.02}{1} \\ &\approx 1.02 \end{aligned}$$

To find the value of the constant $\ln a$ we substitute this value of b and the coordinates of one of the points on the line, say $(1, 3.53)$ into the equation $Y = \ln a + bX$.

$$\begin{aligned} 3.53 &= \ln a + 1.02(1) \\ \ln a &= 3.53 - 1.02 \\ &\approx 2.51 \end{aligned}$$

So, $a = e^{2.51} = 12.3$. Thus the relationship between x and y is given approximately by the equation

$$y = 12.3 \times x^{1.02}.$$

A remark about this solution: yours may be slightly different from this one. Remember that the choice of line of best fit is a subjective one, and that we have rounded all numbers to a couple of decimal places. However, if correct, your answer will not differ from this one by much.

2. A table of the values of x and $Y = \ln y$ is given in Figure 16.

x	0.58	2.10	3.14	4.08	4.52	5.96	7.46	8.72
$Y = \ln y$	-1.08	-0.63	-0.04	0.37	0.46	1.26	1.78	2.30

Figure 16: Values of x and $Y = \ln y$ for exercise 2.2.

The data have been plotted and a line of best fit drawn in Figure 17. The point

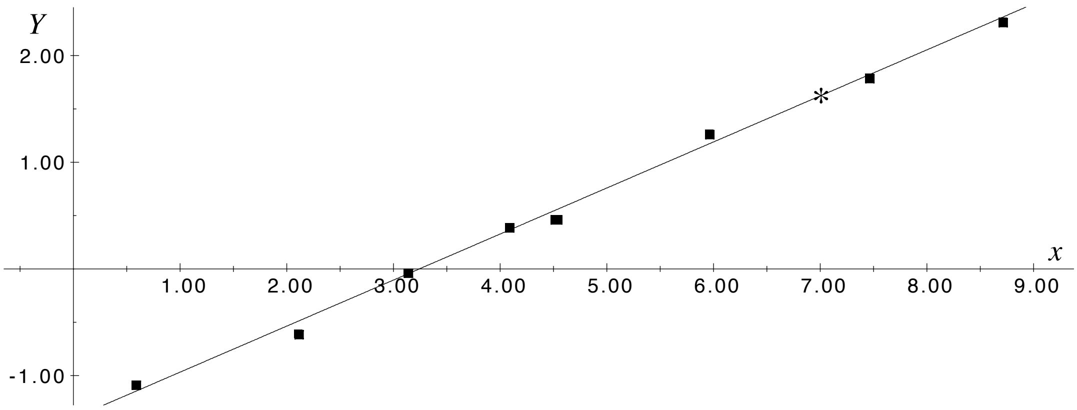

Figure 17: Plot of $Y = \ln y$ against x for exercise 2.2.

$(x, Y) = (3.14, -0.04)$ is on the line. We have marked by an asterix another point on the line, $(7, 1.61)$. We calculate the slope of this line, k, using these two points. In calculating the slope of the line we will take the logarithm of the y-values only, not of the x-values. Taking logarithms to base e of both sides of the equation $y = ae^{kx}$ gives $\ln y = \ln a + kx$, and the slope of the line is k.

$$\begin{aligned} k &\approx \frac{1.61 - (-0.04)}{7 - 3.14} \\ &\approx \frac{1.65}{3.86} \\ &\approx 0.43. \end{aligned}$$

To find the value of the constant $\ln a$ we substitute $k = 0.43$ and the coordinates of one of the points on the line into the equation $Y = \ln a + kx$. Substituting the point $(3.14, -0.04)$ we get

$$\begin{aligned} -0.04 &\approx \ln a + (0.43)(3.14) \\ \ln a &\approx -0.04 - 1.35 \\ &\approx -1.39. \end{aligned}$$

So, $a = e^{-1.39} = 0.25$

The data conform approximately to the relationship $y = 0.25e^{0.43x}$.

3. Plotting $Y = \ln y$ against $X = \ln x$ results in a straight line. (If you plot $Y = \ln y$ against x you get a curve.) Therefore these data conform to a relationship of the form $y = ax^b$. A table of $X = \ln x$ and $Y = \ln y$ is given in Figure 18. A plot of $Y = \ln y$ against $X = \ln x$ has been made in Figure 19, and a line of best fit has been drawn in.

$X = \ln x$	0.56	1.03	1.36	1.84	2.17	2.72	3.24	3.74
$Y = \ln y$	0.66	1.20	1.74	2.16	2.66	3.18	3.76	4.44

Figure 18: Values of X and Y for exercise 2.3.

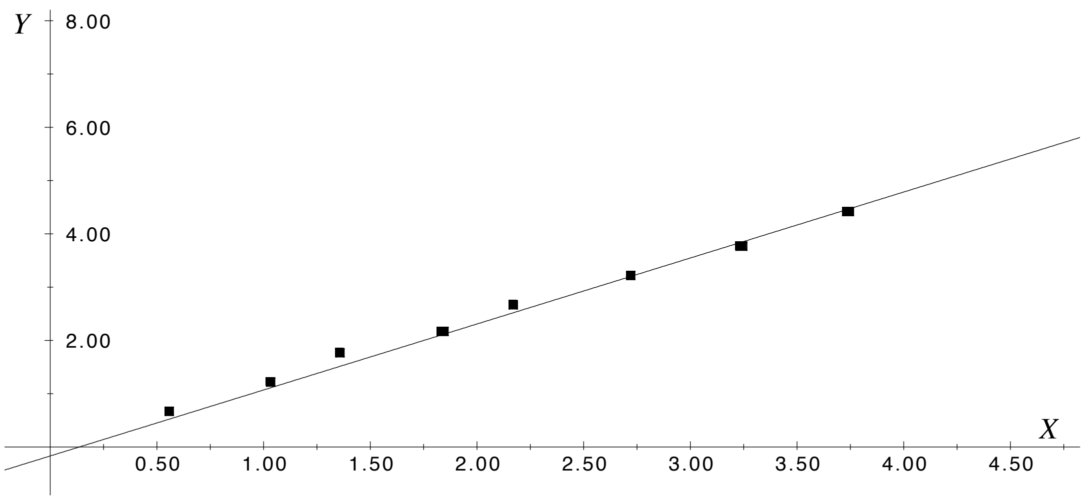

Figure 19: Plot of $Y = \ln y$ against $X = \ln x$ for exercise 2.3.

On the line of best fit we have chosen the points $(2.72, 3.18)$ and $(3.74, 4.44)$ to calculate the constants a and b.

$$\begin{aligned} b &= \frac{4.44 - 3.18}{3.74 - 2.72} \\ &\approx \frac{1.26}{1.02} \\ &\approx 1.24 \end{aligned}$$

To find the constant $\ln a$ we substitute the coordinates of one of the points on the line into the equation $Y = \ln a + bX$. We will use the point (2.72,3.18).

$$\begin{aligned} 3.18 &\approx \ln a + (1.24)(2.72) \\ \ln a &\approx 3.18 - 3.37 \\ &\approx -0.19 \end{aligned}$$

So, $a = e^{-0.19} = 0.82$.

The data conform approximately to the relationship $y = 0.82x^{1.24}$.

4. Plotting $Y = \ln y$ against $X = \ln x$ will result in a curve, whereas plotting $Y = \ln y$ against x results in a straight line. Therefore the data conform approximately to a relationship of the form $y = ae^{kx}$. Figure 20 is a table of x and $Y = \ln y$. Figure 21 shows a plot of $Y = \ln y$ against x and a line of best fit has been drawn.

x	1.12	2.22	3.08	3.94	5.00	6.04	7.14	8.18
$Y = \ln y$	2.90	3.56	4.20	4.64	5.34	5.96	6.54	7.24

Figure 20: Values of x and Y for exercise 2.4.

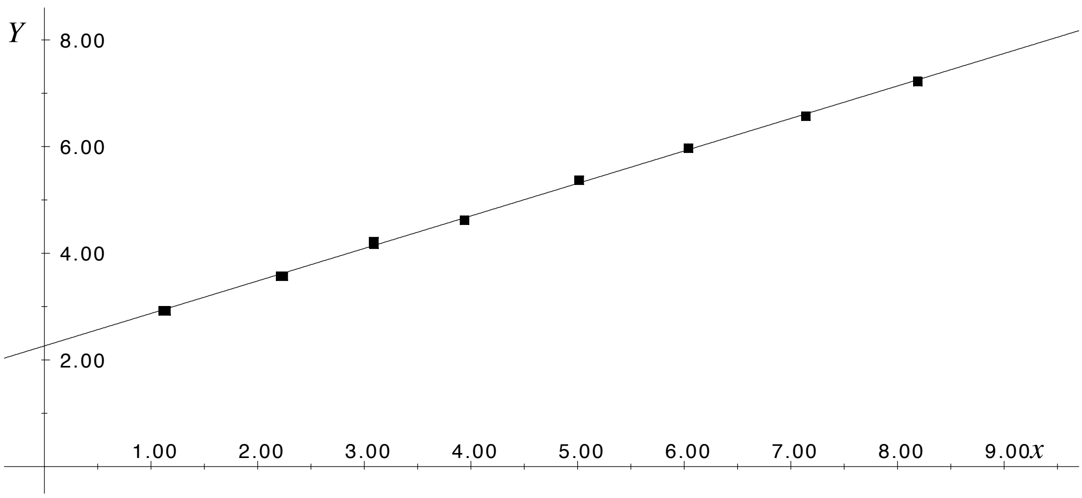

Figure 21: Plot of $Y = \ln y$ against x for exercise 2.4.

On the line of best fit we have chosen the points $(1.12, 2.90)$ and $(8.18, 7.24)$, to calculate the constants a and b,

$$\begin{aligned} k &\approx \frac{7.24 - 2.90}{8.18 - 1.12} \\ &\approx \frac{4.34}{7.06} \\ &\approx 0.61 \end{aligned}$$

Substituting the coordinates of the point $(1.12, 2.90)$ into the equation $Y = \ln a + kx$, we get

$$\begin{aligned} 2.90 &\approx \ln a + (1.12)(0.61) \\ \ln a &\approx 2.90 - 0.68 \\ &\approx 2.22 \\ \text{So,}\quad a &\approx 9.20 \end{aligned}$$

Thus the data conform approximately to the relationship $y = 9.20 \times e^{0.61x}$.

5. Plotting $Y = \ln y$ against x results in a straight line, so the data conform approximately to the relationship $y = ae^{kx}$. The values of x and Y are given in Figure 22. These values and a line of best fit have been drawn in Figure 23. For the calculation of the constants a and b, the points $(3.00, 7.93)$ and $(5.0, 7.00)$ have been selected on the line of best fit, and these points are each marked with an asterix.

x	1.00	1.98	3.28	4.58	6.00	6.90	7.82	9.00
$Y = \ln y$	8.91	8.31	7.85	7.10	6.48	6.14	5.61	5.19

Figure 22: Values of x and $Y = \ln y$ for exercise 2.5.

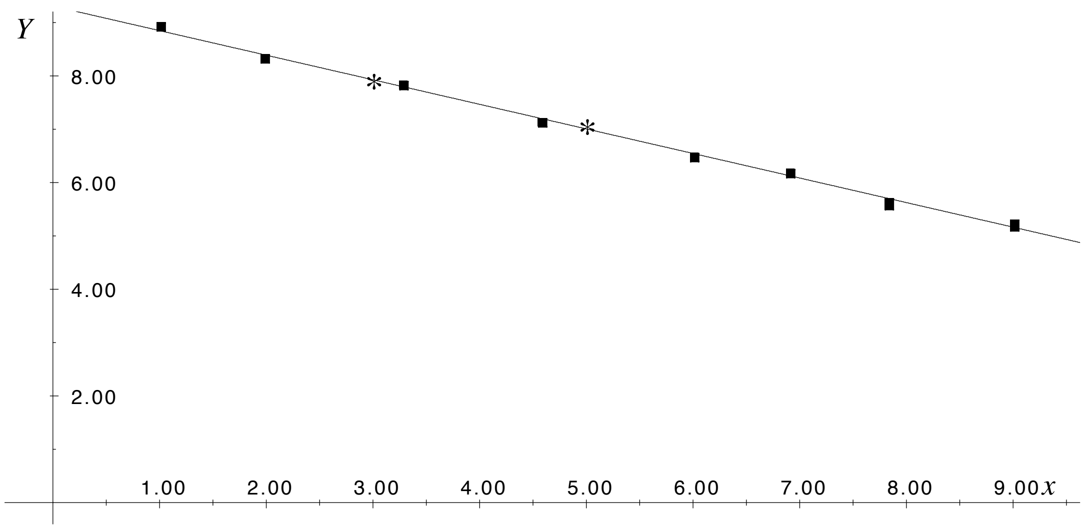

Figure 23: Plot of Y against x for exercise 2.5.

$$\begin{aligned} k &\approx \frac{7.00 - 7.93}{5.00 - 3.00} \\ &\approx \frac{-0.93}{2.00} \\ &\approx -0.47 \end{aligned}$$

Substituting the coordinates of the point $(5.00, 7.00)$ into the equation $Y = \ln a + kx$ gives us

$$7.00 \approx \ln a + (-0.47)(5.00)$$

$$\begin{aligned} \ln a &\approx 7.00 + 2.35 \\ &\approx 9.35 \\ \text{So,} \quad a &\approx 11499 \end{aligned}$$

Thus the data conform approximately to the relationship $y = 11499e^{-0.47x}$.

6. Plotting $Y = \ln y$ against $X = \ln x$ results in a straight line whereas plotting $Y = \ln y$ against x will result in a curve. Therefore this data conforms to a relationship of the form $y = ax^b$. A table of the values of $Y = \ln y$ and $X = \ln x$ is given in Figure 24. A plot of $Y = \ln y$ against $X = \ln x$ has been made in Figure 25, and a line of best fit drawn in.

$X = \ln x$	-1.25	-0.98	-0.55	-0.19	0.21	0.74
$Y = \ln y$	2.80	3.48	4.24	5.08	5.76	7.00

Figure 24: X and Y values for exercise 2.6.

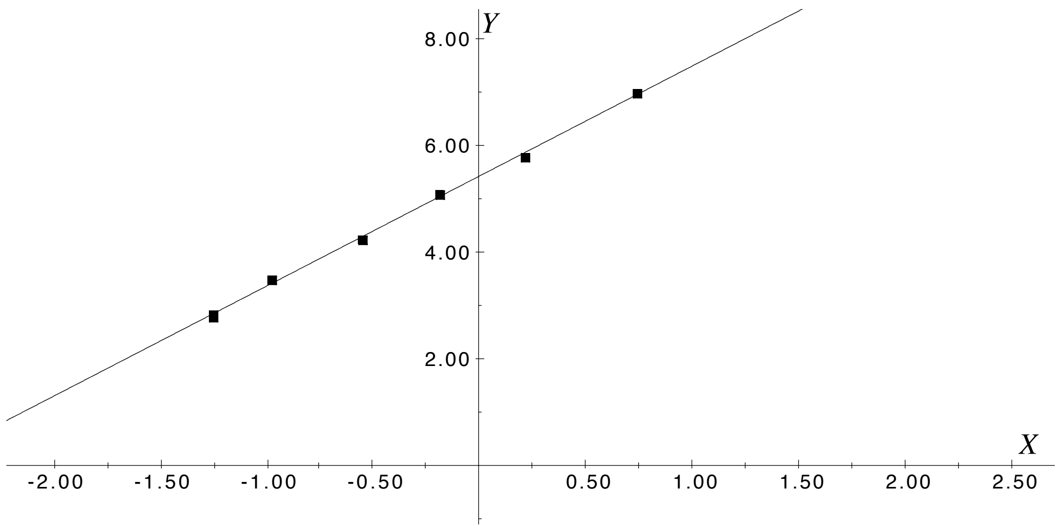

Figure 25: Plot of Y against X for exercise 2.6.

On the line of best fit we have chosen the points $(-1.25, 2.80)$ and $(0.74, 7.00)$ to calculate the constants a and b.

$$\begin{aligned} b &= \frac{7.00 - 2.80}{0.74 - (-1.25)} \\ &\approx \frac{4.20}{1.99} \\ &\approx 2.11. \end{aligned}$$

To find the constant $\ln a$ we substitute the coordinates of one of the points on the line into the equation $Y = \ln a + bX$. We will use the point (0.74, 7.00).

$$\begin{aligned} 7.00 &\approx \ln a + (2.11)(0.74) \\ \ln a &\approx 7.00 - 1.56 \\ &\approx 5.44 \\ \text{So,} \quad a &\approx 230.44. \end{aligned}$$

The data conform approximately to the relationship $y = 230x^{2.11}$.

5.3 Solutions to Exercises from Section 3

1. $\frac{d(\ln 2x^3)}{dx} = \frac{d(2x^3)}{dx} \times \frac{1}{2x^3} = \frac{6x^2}{2x^3} = \frac{3}{x}$

2. $\frac{d(e^{x^7})}{dx} = \frac{d(x^7)}{dx} \times e^{x^7} = 7x^6 e^{x^7}$

3. $\frac{d(\ln 11x^7)}{dx} = \frac{d(11x^7)}{dx} \times \frac{1}{11x^7} = \frac{77x^6}{11x^7} = \frac{7}{x}$

4. $\frac{d(e^{x^2+x^3})}{dx} = \frac{d(x^2+x^3)}{dx} \times e^{x^2+x^3} = (2x+3x^2)e^{x^2+x^3}$

5.

$$\begin{aligned}\frac{d(e^{7x^{-2}})}{dx} &= \frac{d(7x^{-2})}{dx} \times e^{7x^{-2}} \\ &= (-14x^{-3}) \times e^{7x^{-2}} \\ &= \frac{-14e^{7x^{-2}}}{x^3}\end{aligned}$$

6.

$$\begin{aligned}\frac{d\ln(e^x + x^3)}{dx} &= \frac{d(e^x + x^3)}{dx} \times \frac{1}{e^x + x^3} \\ &= (e^x + 3x^2) \times \frac{1}{e^x + x^3} \\ &= \frac{e^x + 3x^2}{e^x + x^3}\end{aligned}$$

7.

$$\begin{aligned}\frac{d(e^{\frac{2x}{x^2+1}})}{dx} &= \frac{d(\frac{2x}{x^2+1})}{dx} \times e^{\frac{2x}{x^2+1}} \\ &= \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2} \times e^{\frac{2x}{x^2+1}} \\ &= \frac{2-2x^2}{(x^2+1)^2} \times e^{\frac{2x}{x^2+1}}\end{aligned}$$

8.

$$\begin{aligned}\frac{d(\ln(\frac{5x-2}{x^2+3}))}{dx} &= \frac{d(\frac{5x-2}{x^2+3})}{dx} \times \frac{1}{\frac{5x-2}{x^2+3}} \\ &= \frac{(x^2+3)(5) - (5x-2)(2x)}{(x^2+3)^2} \times \frac{x^2+3}{5x-2} \\ &= \frac{15+4x-5x^2}{(x^2+3)^2} \times \frac{x^2+3}{5x-2} \\ &= \frac{15+4x-5x^2}{(x^2+3)(5x-2)}\end{aligned}$$

9.

$$\begin{aligned}
\frac{d(\ln(e^x x^8))}{dx} &= \frac{d(e^x x^8)}{dx} \times \frac{1}{e^x x^8} \\
&= ((e^x)(8x^7) + (x^8)(e^x)) \times \frac{1}{e^x x^8} \\
&= \frac{e^x(x^8 + 8x^7)}{e^x x^8} \\
&= \frac{x^8 + 8x^7}{x^8}
\end{aligned}$$

10. $\int \frac{2}{2x+1}\, dx = \ln|2x+1| + C$

11. $\int 2xe^{x^2+2}\, dx = e^{x^2+2} + C$

12. $\int 3x^2 e^{x^3}\, dx = e^{x^3} + C$

13. $\int \frac{2x+2}{x^2+2x+7}\, dx = \ln|x^2 + 2x + 7| + C$

14. $\int \frac{x+2}{x^2+4x+1}\, dx = \frac{1}{2}\int \frac{2x+4}{x^2+4x+1}\, dx = \frac{1}{2}\ln|x^2 + 4x + 1| + C$

15. $\int e^{-2x}\, dx = -\frac{1}{2}e^{-2x} + C$

16. $\int \frac{1}{x^3}e^{-\frac{1}{x^2}}\, dx = \frac{1}{2}\int (2x^{-3})e^{-\frac{1}{x^2}}\, dx = \frac{1}{2}e^{-\frac{1}{x^2}} + C$

17. $\int \frac{1}{1-x}\, dx = -\ln(1-x) + C = \ln(\frac{1}{1-x}) + C$

18. $\int \cos x e^{\sin x}\, dx = e^{\sin x} + C$

19.

$$\begin{aligned}
f(x) &= \frac{x^7 \cos x}{(x+1)^2} \\
\ln f(x) &= 7\ln x + \ln(\cos x) - 2\ln(x+1) \\
\frac{f'(x)}{f(x)} &= \frac{7}{x} - \frac{\sin x}{\cos x} - \frac{2}{x+1} \\
f'(x) &= f(x)\left(\frac{7}{x} - \frac{\sin x}{\cos x} - \frac{2}{x+1}\right) \\
&= \frac{x^7 \cos x}{(x+1)^2}\left(\frac{7}{x} - \frac{\sin x}{\cos x} - \frac{2}{x+1}\right)
\end{aligned}$$

20.

$$\begin{aligned}
f(x) &= 3^{-x} \\
\ln f(x) &= (-x)\ln 3 \\
\frac{f'(x)}{f(x)} &= (-1)\ln 3 \\
f'(x) &= f(x)(-\ln 3) \\
&= -(\ln 3)3^{-x}
\end{aligned}$$

21.

$$\begin{aligned}
f(x) &= x^{-3}(x^2+3x+1)^4 5^{-x} \\
\ln f(x) &= -3\ln x + 4\ln(x^2+3x+1) - x\ln 5 \\
\frac{f'(x)}{f(x)} &= -\frac{3}{x} + 4\frac{2x+3}{x^2+3x+1} - \ln 5 \\
f'(x) &= f(x)\left(-\frac{3}{x} + 4\frac{2x+3}{x^2+3x+1} - \ln 5\right) \\
&= x^{-3}(x^2+3x+1)^4 5^{-x}\left(-\frac{3}{x} + 4\frac{2x+3}{x^2+3x+1} - \ln 5\right)
\end{aligned}$$

22. $\frac{d(\ln 3x^4)}{dx} = \frac{12x^3}{3x^4} = \frac{4}{x}$

23. $\frac{d(e^{2x^9})}{dx} = 18x^8e^{2x^9}$

24. $\frac{d(\ln 3x^6)}{dx} = \frac{18x^5}{3x^6} = \frac{6}{x}$

25. $\frac{d(e^{x^5-x^2})}{dx} = (5x^4-2x)e^{x^5-x^2}$

26.

$$\begin{aligned}
\frac{d(\ln 4x^{-6})}{dx} &= \frac{-24x^{-7}}{4x^{-6}} \\
&= \frac{-6}{x}
\end{aligned}$$

27.

$$\begin{aligned}
\frac{d(\ln(e^{x^2}+2x^4))}{dx} &= \frac{d(e^{x^2}+2x^4)}{dx} \times \frac{1}{e^{x^2}+2x^4} \\
&= (2xe^{x^2}+8x^3) \times \frac{1}{e^{x^2}+2x^4} \\
&= \frac{2xe^{x^2}+8x^3}{e^{x^2}+2x^4}
\end{aligned}$$

28.

$$\begin{aligned}
f(x) &= \frac{x^4(x+x^2)^3}{(x-1)^2} \\
\ln f(x) &= 4\ln x + 3\ln(x+x^2) - 2\ln(x-1) \\
\frac{f'(x)}{f(x)} &= \frac{4}{x} + 3\frac{1+2x}{x+x^2} - \frac{2}{x-1} \\
f'(x) &= f(x)\left(\frac{4}{x} + 3\frac{1+2x}{x+x^2} - \frac{2}{x-1}\right) \\
&= \frac{x^4(x+x^2)^3}{(x-1)^2}\left(\frac{4}{x} + 3\frac{1+2x}{x+x^2} - \frac{2}{x-1}\right)
\end{aligned}$$

29.

$$\begin{aligned} f(x) &= 4^x e^{x^2} \\ \ln f(x) &= x\ln 4 + x^2 \ln e \\ \frac{f'(x)}{f(x)} &= \ln 4 + 2x \\ f'(x) &= f(x)(\ln 4 + 2x) \\ &= (\ln 4 + 2x)4^x e^{x^2} \end{aligned}$$

30.

$$\begin{aligned} f(x) &= x^5(x^5+3x^2+1)^3 2^{-x} \\ \ln f(x) &= 5\ln x + 3\ln(x^5+3x^2+1) - x\ln 2 \\ \frac{f'(x)}{f(x)} &= \frac{5}{x} + 3\frac{5x^4+6x}{x^5+3x^2+1} - \ln 2 \\ f'(x) &= f(x)\left(\frac{5}{x} + 3\frac{5x^4+6x}{x^5+3x^2+1} - \ln 2\right) \\ &= x^5(x^5+3x^2+1)^3 2^{-x}\left(\frac{5}{x} + 3\frac{5x^4+6x}{x^5+3x^2+1} - \ln 2\right) \end{aligned}$$

31. $\int 4x^3 e^{x^4}\, dx = e^{x^4} + C$

32. $\int \frac{3x^2+7}{x^3+7x+7}\, dx = \ln|x^3+7x+7| + C$

33. $\int \frac{1}{2}x^3 e^{x^4+1}\, dx = \frac{1}{8}\int 4x^3 e^{x^4+1}\, dx = \frac{1}{8}e^{x^4+1} + C$

34. $\int \frac{\sin x}{\cos x+5}\, dx = -\int \frac{-\sin x}{\cos x+5}\, dx = -\ln|\cos x + 5| + C$

35. $\int \frac{\sqrt{\ln x}}{x}\, dx = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$

36. $\int \frac{1}{x^4}e^{-\frac{1}{x^3}}\, dx = \frac{1}{3}\int 3x^{-4}e^{-\frac{1}{x^3}}\, dx = \frac{1}{3}e^{-\frac{1}{x^3}} + C$

37. $\int e^x e^{e^x}\, dx = e^{e^x} + C$

38. $\int \frac{5}{x}\, dx = 5\ln|x| + C$

39. $\int xe^{-x^2}\, dx = -\frac{1}{2}\int(-2x)e^{-x^2}\, dx = -\frac{1}{2}e^{-x^2} + C$

5.4 Solutions to Exercises from Section 4

1. **a.** Since $2.1 > 0$, the quantity $P(t) = 1300e^{2.1t}$ is increasing with time.

b. $P(0) = 1300e^{2.1\times 0} = 1300$.

c. The value of P after 1.5 seconds is $P(1.5) = 1300e^{2.1\times 1.5} \approx 30337$.

d. When $P(t) = 2000$ then

$$\begin{aligned} 2000 &= 1300e^{2.1t} \\ e^{2.1t} &= \frac{2000}{1300} = \frac{20}{13} \\ 2.1t &= \ln\frac{20}{13} \\ t &= \frac{1}{2.1}\ln\frac{20}{13} \\ &\approx 0.205 \end{aligned}$$

2. a. We know that $W(t)$ is given by an expression of the form

$$W(t) = W_0e^{kt}.$$

where W_0 is the initial value of the quantity W, that is $W_0 = 22.5$. We also know that $W(t) = 15.5$ when $t = 3$, and so

$$\begin{aligned} 15.5 &= 22.5e^{3k} \\ e^{3k} &= \frac{15.5}{22.5} \\ 3k &= \ln\frac{15.5}{22.5} \\ k &= \frac{1}{3}\ln\frac{15.5}{22.5} \\ &\approx -0.124 \end{aligned}$$

Thus an expression for $W(t)$ is $W(t) = 22.5e^{-0.124t}$.

b. The quantity will have decreased to 10.0 when $W(t) = 10$. That is, when

$$\begin{aligned} 10.0 &= 22.5e^{-0.124t} \\ e^{-0.124t} &= \frac{10.0}{22.5} \\ -0.124t &= \ln\frac{10.0}{22.5} \\ t &= \frac{1}{-0.124}\ln\frac{10.0}{22.5} \\ &\approx 6.5\text{hours} \end{aligned}$$

3. The initial amount of plutonium released was 400g so when $t = 0$, $P(0) = P_0 = 400$.

The equation for decay is $P(t) = 400e^{kt}$.

Plutonium has a half life of 243,000 years, so when $t = 243000$, $P = \frac{1}{2}400$, so

$$\begin{aligned} P(243000) = 200 &= 400e^{k(243000)} \\ e^{k(243000)} &= \frac{1}{2} \\ 243000k &= \ln\frac{1}{2} \\ \text{So,} \quad k &= \frac{\ln\frac{1}{2}}{243000}. \end{aligned}$$

Therefore,

$$P(t) = 400e^{\frac{\ln\frac{1}{2}}{243000}t}.$$

The time taken for 400g to decay to $6.4 \times 10^{-7}g$ is given by the equation

$$\begin{aligned}
6.4 \times 10^{-7} &= 400e^{\frac{\ln\frac{1}{2}}{243000}t} \\
e^{\frac{\ln\frac{1}{2}}{243000}t} &= \frac{6.4 \times 10^{-7}}{400} \\
\frac{\ln\frac{1}{2}}{243000}t &= \ln\frac{6.4 \times 10^{-7}}{400} \\
t &= \frac{243000}{\ln\frac{1}{2}}\ln\frac{6.4 \times 10^{-7}}{400} \\
&\approx 7.1 \times 10^6\text{years.}
\end{aligned}$$

4. From the expression $P'(t) = 4.5P(t)$ we obtain

$$\begin{aligned}
\int \frac{P'}{P}dt &= \int 4.5dt \\
\ln P &= 4.5t + C \\
P(t) &= e^{4.5t+C} \\
&= e^C e^{4.5t}
\end{aligned}$$

where C is an arbitrary constant. Since we know that $P(0) = 480$, the expression must be

$$P(t) = 480e^{4.5t}.$$

5. **a.** The quantity grows exponentially, so it behaves according to an equation of the form $A(t) = A_0e^{kt}$. We are told that

$$\begin{aligned}
A(2) &= 153 \\
A(3) &= 247
\end{aligned}$$

Substituting into the equation for $A(t)$, we get

$$\begin{aligned}
153 &= A_0e^{2k} \\
247 &= A_0e^{3k}.
\end{aligned}$$

This is a pair of equations which can be solved simultaneously to find the constants A_0 and k. Dividing the second by the first,

$$\begin{aligned}
\frac{247}{153} &= \frac{A_0e^{3k}}{A_0e^{2k}} \\
&= e^k \\
k &= \ln\frac{247}{153} \\
&\approx 0.479.
\end{aligned}$$

We can now substitute this value of k into either of the pair of equations to obtain a value for A_0. If we choose the first of the pair then

$$A_0 \approx \frac{153}{e^{2\times 0.479}} \approx 58.7$$

.

b. From the previous part of the solution, $A(t) \approx 58.7e^{.479t}$.

c. $A(7) \approx 58.7e^{0.479\times 7} \approx 1678$.

d. The time t when the quantity A is equal to 2000 is given by the equation

$$2000 = 58.7e^{0.479t}$$

$$\begin{aligned}
e^{0.479t} &= \frac{2000}{58.7} \\
0.479t &= \ln\frac{2000}{58.7} \\
t &= \frac{1}{0.479}\ln\frac{2000}{58.7} \\
&\approx 7.37 \quad \text{minutes.}
\end{aligned}$$

6. a. Since the quantity grows exponentially, and because the initial quantity is 47.9, we know that it behaves according to a relationship of the form $B(t) = 47.9e^{kt}$. The other piece of information we are given is

$$\begin{aligned}
B(6.7) = 47.9e^{k\times 6.7} &= 102, \qquad \text{which means that} \\
e^{k\times 6.7} &= \frac{102}{47.9} \\
k \times 6.7 &= \ln\frac{102}{47.9} \\
k &= \frac{1}{6.7}\ln\frac{102}{47.9} \\
&\approx 0.113.
\end{aligned}$$

The quantity B behaves according to the equation $B(t) = 47.9e^{0.113t}$.

b. $B(t)$ has increased to 500 when

$$\begin{aligned}
B(t) = 47.9e^{0.113t} &= 500 \\
e^{0.113t} &= \frac{500}{47.9} \\
0.113t &= \ln\frac{500}{47.9} \\
t &= \frac{1}{0.113}\ln\frac{500}{47.9} \\
&\approx 20.76.
\end{aligned}$$

The quantity will have increased to 500 after approximately 20.76 minutes.

7. a. Radium decays exponentially, so it decays according to a relationship of the form

$$R(t) = R_0e^{kt}.$$

We are told that, no matter what R_0 is, after 1466 years the sample will have decayed to $R_0/2$. In symbols

$$\begin{aligned}
R(1466) = \frac{R_0}{2} &= R_0e^{k\times 1466} \\
e^{k\times 1466} &= \frac{1}{2} \\
k \times 1466 &= \ln\frac{1}{2} \\
k &= \frac{1}{1466}\ln\frac{1}{2} \\
&\approx -4.73 \times 10^{-4}.
\end{aligned}$$

Thus the equation governing the decay of radium is

$$R(t) = R_0e^{\frac{\ln\frac{1}{2}}{1466}t}.$$

b. The amount of radium decays to one fifth of its original level when

$$
\begin{aligned}
e^{\frac{\ln \frac{1}{2}}{1466}t} &= \frac{1}{5} \\
\frac{\ln \frac{1}{2}}{1466}t &= \ln \frac{1}{5} \\
t &= \frac{1466}{\ln \frac{1}{2}} \ln \frac{1}{5} \\
&\approx 3404 \quad \text{years.}
\end{aligned}
$$

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